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Systematic singular triangulations of all Seifert manifolds

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本論文は, 投稿予定の論文をベースにしていますので, 1. Introduction 以外は英語となっていますが, ご容赦願います.

1 Introduction

任意の閉3次元多様体には, 頂点が1つである singular triangulations が存在する（[1] を参照）. これを *one-vertex triangulation* と呼ぶことにする. 本報告では, 全ての Seifert manifold M に対して, その表示 $S(F_g, b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_{n+1}, \beta_{n+1}))$ から, M の one-vertex triangulation を構成する方法を述べる.

この構成方法を用いると, Seifert manifold M の one-vertex triangulation は, $\{L, R, \bar{L}, \bar{R}\}$ の有限列（word）で表される（Section 8 を参照）. 特に, レンズ空間 $L(p, q)$ は, q/p の連分数展開に関係した $\{L, R\}$ の word で表される（Section 6 を参照）.

この構成方法は次のような段階に分かれている. 始めに, $\{L, R, \bar{L}, \bar{R}\}$ 上の word に対して, 辺に向きとラベルが与えられた 3-regular graph G を定義する（Section 2, 8 を参照）. 次に, B^3 の境界に埋め込まれた G が, ∂B^3 の同一視写像 f_G を導くことを示す. 最後に, 任意の Seifert manifold M に対して, $\{L, R, \bar{L}, \bar{R}\}$ 上の word が存在して, この word によって定義される graph G が導く同一視写像 f_G が, B^3/f_G が M と同相であり, $(\partial B^3)/f_G$ が M の special spine となることを示す. special spine の dual cell complex は one-vertex triangulation であるから, M の one-vertex triangulation が得られたことになる（Section 9 を参照）.

歴史的には, 3-regular graph G から導かれる ∂B^3 上の同一視写像を用いて, 3次元多様体 $M\,(\cong B^3/f_G)$ を表示することは新しいことではない. 例えば, quaternionic space の 3-regular graph による表示は文献 [2] の図 19 にある. これらの表示方法に対して, 本報告で示す表示方法は次の2点において, 新しく, 有効である.

1つは, M の fiber 構造が明らかであることである. 任意の M に対して, $M \cong B^3/f_G$ となる 3-regular graph G を構成したのだが, この G には, *E-cycle* と呼ばれる cycle e が存在する. E-cycle の定義は [3]-[6] にある. この E-cycle e の性質の1つとして, $\mathbb{R}^3$ 内の単位球 B^3 の境界に, $e = \partial B^3 \cap \{z = 0\}$ となるように G を埋め込んだとき, ベクトル場 $\partial/\partial z$ で生成される flow によって M の fiber structure が1つ定まる. 例えば, Figure 1 は quaternionic space の表示であり, E-cycle は $X_1X_2Y_1Y_2Z_1Z_2$ である.

もう1つは, M の Dehn surgery を表示しやすいことである. 任意の Seifert manifold M に対して, M の singular fiber に沿った Dehn surgery によって得られる manifold M' は, $M \cong B^3/f_G$ となるグラフ G の一部を取り替えたグラフ G' によって, $M' \cong B^3/f_{G'}$ となる. 例を挙げると, Figure 2 のグラフによって表されている manifold は, Figure 1 のグラフによって表されている manifold を, $(2,1)$ type singular fiber に沿った -1 Dehn surgery した manifold である.

また, 本報告での表示方法を用いると, fibered solid, $(S^2 - (D^2 \cup D^2 \cup D^2)) \times S^1$, $(S^1 \times S^1 - D^2) \times S^1$ に対して, ベクトル値 Turaev-Viro invariant を定義することができ, Seifert manifold の Turaev-Viro invariant が, これらのベクトルの内積によって計算される（[7] を

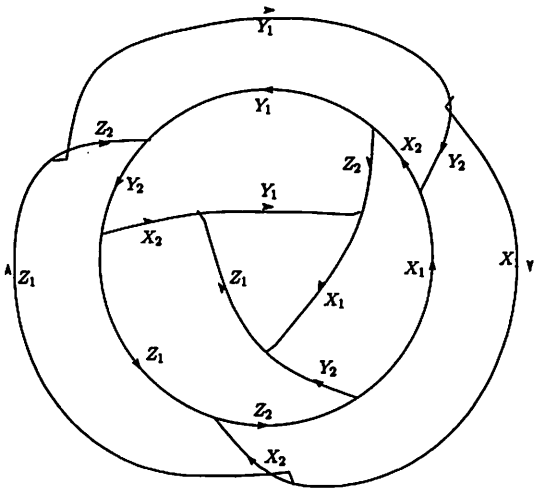

Figure 1: Graphical representation of quaternionic space

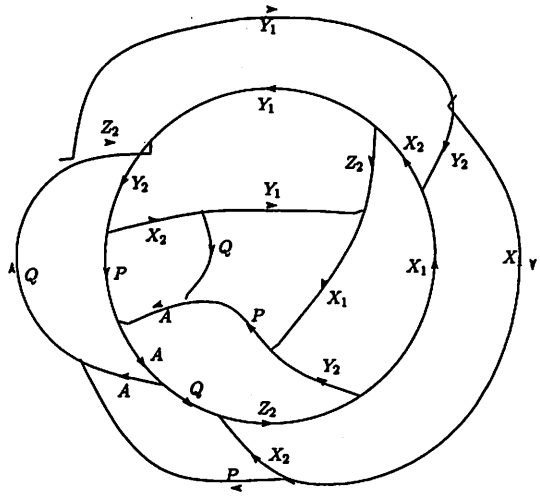

Figure 2: Dehn surgery along a singular fiber

参照）．また，3次元多様体に対する新たな complexity であるブロック数が定義できる（[8] を参照）．

2 Definition of solid torus $T_{w(L,R)}$

In this section, we define a *word diagram* for a word on the letters $\{L, R, \bar{L}, \bar{R}\}$ denoted by $w(L, R)$. After we define a $w(L, R)$-*solid torus.*

The the directed labeled 3-regular graphs shown in Figure 3, Figure 4, Figure 5, Figure 6 and Figure 7 are called L - *ring*, $\bar{L}$ - *ring*, R - *ring*, $\bar{R}$ - *ring* and ϕ - *diagram* respectively. Let W be a word $X_1X_2\cdots X_n$, where $X_i \in \{L, R, \bar{L}, \bar{R}\}$. We define a W-diagram inductively.

In the case $n = 0$, W-diagram is defined as ϕ-diagram. Then we define a W-diagram in the case $n = 1$.

Definition 2.1 *For a word $W = X$, where $X \in \{L, R, \bar{L}, \bar{R}\}$, the W-diagram is defined by the following two steps:*

1. *Identify the circle $\alpha\,\beta\,\gamma\,\alpha^{-1}\,\beta^{-1}\,\gamma^{-1}$ of the ϕ - diagram and the circle $\alpha'\,\beta'\,\gamma'\,\alpha'^{-1}\,\beta'^{-1}\,\gamma'^{-1}$ of the X - ring such that the directed edges α, β, γ and α', β', γ' are identified respectively.*

2. *Delete the edge α, β, γ of the ϕ - diagram and α', β', γ' of an X - ring.*

An example in the case $X = L$ is shown in Figure 8. Then, we define a W-diagram in the case $n > 1$.

Definition 2.2 *Let W be a word $X_1X_2\cdots X_n$, where $X_i \in \{L, R, \bar{L}, \bar{R}\}$ $(1 \le i \le n)$. Suppose the W - diagram is defined. Then, for a word $X_{n+1} \in \{L, R, \bar{L}, \bar{R}\}$, the WX_{n+1} - diagram is defined by the following two steps:*

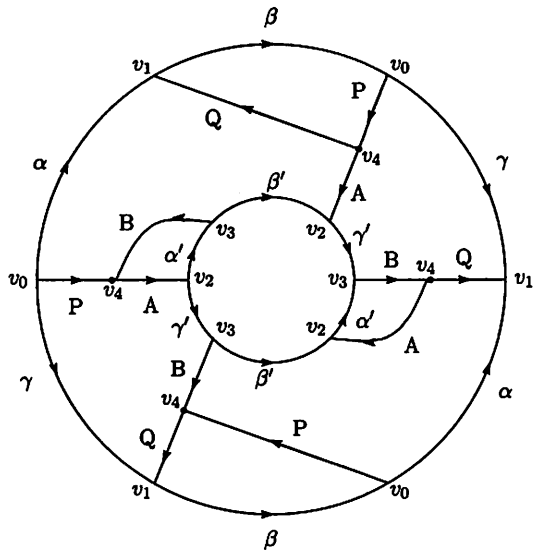

Figure 3: L-ring

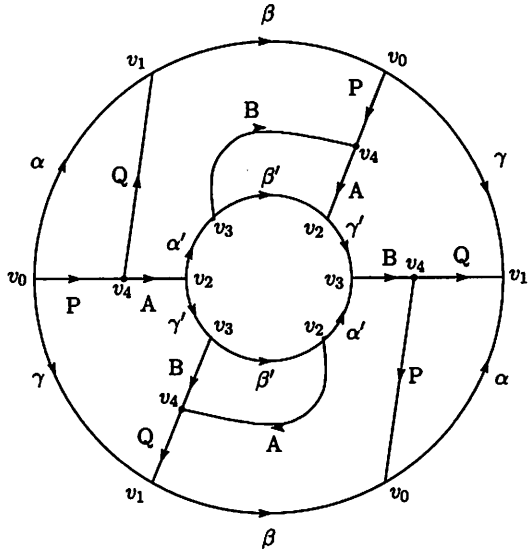

Figure 4: $\bar{L}$-ring

1. *Identify the circle* $\alpha\,\beta\,\gamma\,\alpha^{-1}\,\beta^{-1}\,\gamma^{-1}$ *of the* W*-diagram and the circle* $\alpha'\,\beta'\,\gamma'\,\alpha'^{-1}\,\beta'^{-1}\,\gamma'^{-1}$ *of the* X_{n+1}*-ring such that the directed edges* α, β, γ *and* α', β', γ' *are identified respectively.*

2. *Delete the edge* α, β, γ *of the* W*-diagram and* α', β', γ' *of an* X_{n+1}*-ring.*

We call such a diagram a *word diagram* and denote them by $w(L,R)$-diagram.

We assume that $w(L,R)$-diagram is embedded in $S^2\,(\cong \partial B^3)$. Then the circle $\alpha\,\beta\,\gamma\,\alpha^{-1}\,\beta^{-1}$
γ^{-1} of $w(L,R)$-diagram separate S^2 to two discs.

Notation 2.3 *The disc which contains the edges of* $w(L,R)$*-diagram is denoted by* $D_{w(L,R)}$. *Another disc is denoted by* $E_{w(L,R)}$.

For example, D_L is $\mathcal{A}\cup\mathcal{A}'\cup\mathcal{B}\cup\mathcal{B}'\cup\mathcal{C}\cup\mathcal{C}'\cup\mathcal{D}\cup\mathcal{D}'$ in Figure 9.

Let $f_{w(L,R)}$ be an identification map on $S^2\,(\cong \partial B^3)$ induced from the identifying the directed labeled edges of $w(L,R)$-diagram. We give an example f_L: the face $\mathcal{A}$ and $\mathcal{A}'$ are identified by f_L because both faces is bounded by the circle PQ of the L-diagram, see Figure 9.

We consider the manifold $B^3/f_{w(L,R)}$. In Proposition 2.6, we will show that $B^3/f_{w(L,R)}$ is solid torus for any word $w(L,R)$. Before discussion, we prepare some lemmas.

Lemma 2.4 *The manifold* B^3/f_ϕ *collapses to* S^1, *where* f_ϕ *is the identification map in the case* $w(L,R)=\phi$.

Proof. By the definition of f_ϕ, the boundary of B^3/f_ϕ is E_ϕ/f_ϕ. Thus, the manifold B^3/f_ϕ collapses to the cell complex D_ϕ/f_ϕ from its boundary, where D_ϕ is shown Figure 10. And

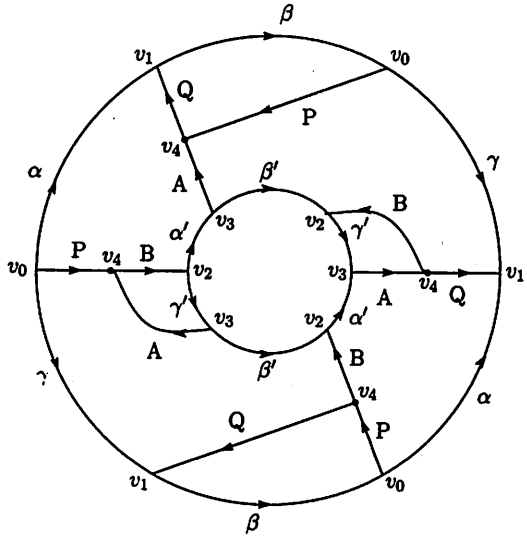

Figure 5: R-ring

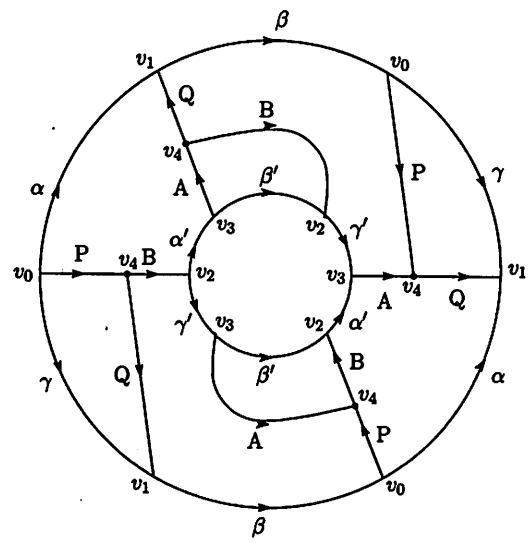

Figure 6: $\bar{R}$-ring

the cell complex D_ϕ/f_ϕ collapses to a cell complex D'_ϕ/f_ϕ, where D'_ϕ is shown Figure 11. Similarly, the cell complex D'_ϕ/f_ϕ collapses to a loop B/f_ϕ, where B is the edge shown in Figure 11. Thus, B^3/f_ϕ collapses to S^1. ∎

Lemma 2.5 *If the manifold $B^3/f_{w(L,R)}$ collapses to S^1, then the manifold $B^3/f_{w(L,R)X}$ collapses to S^1, where $X \in \{L, R, \bar{L}, \bar{R}\}$.*

Proof.

1. The case $X = L$

   The manifold $B^3/f_{w(L,R)L}$ is collapsed to a cell complex $D_{w(L,R)L}/f_{w(L,R)L}$ from its boundary, where $D_{w(L,R)L}$ is shown Figure 12. It collapses to the cell complex $D'_{w(L,R)L}/f_{w(L,R)L}$ from its boundary, where $D'_{w(L,R)L}$ is shown Figure 13. Similarly, the cell complex $D'_{w(L,R)L}/f_{w(L,R)L}$ collapses to the cell complex $D''_{w(L,R)L}/f_{w(L,R)L}$, where $D''_{w(L,R)L}$ is shown Figure 14. At last, the cell complex $D''_{w(L,R)L}/f_{w(L,R)L}$ collapses to a cell complex $D'''_{w(L,R)L}/f_{w(L,R)L}$, where $D'''_{w(L,R)L}$ is shown Figure 15.

   By definition of $f_{w(L,R)}$, $D'''_{w(L,R)L}/f_{w(L,R)L}$ is $D'''_{w(L,R)}/f_{w(L,R)}$. By assumption of induction, $D'''_{w(L,R)}/f_{w(L,R)}$ collapses to S^1. Thus, $M_{w(L,R)L}$ collapses to S^1.

2. The case $X \in \{R, \bar{L}, \bar{R}\}$

   By collapsing $B^3_{w(L,R)X}/f_{w(L,R)X}$ from its boundary similar to the case 1, the proof completes. ∎

Proposition 2.6 *For any word $w(L, R)$, the manifold $B^3/f_{w(L,R)}$ is homeomorphic to a solid torus.*

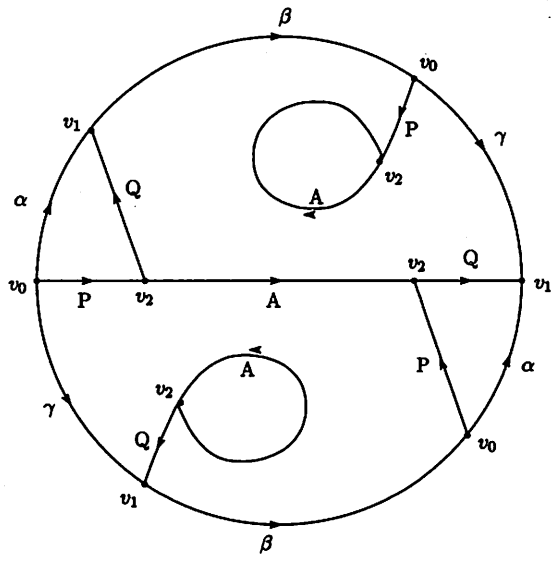

Figure 7: ϕ-diagram

Proof. By Lemma 2.4 and Lemma 2.5, $B^3/f_{w(L,R)}$ collapses to S^1. Thus, $B^3/f_{w(L,R)}$ is homeomorphic to a solid torus or a solid Klein bottle. By the definition of ϕ-diagram and $L, R, \bar{L}, \bar{R}$-ring, the manifold $B^3/f_{w(L,R)}$ is orientated. Thus, $B^3/f_{w(L,R)}$ is solid torus. ∎

The solid torus $B^3/f_{w(L,R)}$ is called a $w(L,R)$ *solid torus* and denoted by $T_{w(L,R)}$.

Proposition 2.7 *For any solid tours $T_{w(L,R)}$, the θ-curve shown in Figure 16 is embedded in $\partial T_{w(L,R)}$ such that $\partial T_{w(L,R)} \setminus (\alpha \cup \beta \cup \gamma)$ is homeomorphic to an open 2-disc.*

Proof. Recall $E_{w(L,R)}$, see Notation 2.3. By definition of $T_{w(L,R)}$, the boundary of $T_{w(L,R)}$ is $E_{w(L,R)} / f_{w(L,R)}$. Thus, θ-curve is embedded in $\partial T_{w(L,R)}$ such that $\partial T_{w(L,R)} \setminus (\alpha \cup \beta \cup \gamma)$ is homeomorphic to an open 2-disc. ∎

Notation 2.8 *The edges α, β, γ of the θ-curve embedded in $\partial T_{w(L,R)}$ are denoted by $\alpha_{w(L,R)}, \beta_{w(L,R)}, \gamma_{w(L,R)}$ respectively. Then, $\alpha_{w(L,R)}\beta_{w(L,R)}, \beta_{w(L,R)}\gamma_{w(L,R)}, \gamma_{w(L,R)}\alpha_{w(L,R)}^{-1}$ are loops embedded in $\partial T_{w(L,R)}$. We denote them by $x_{w(L,R)}, y_{w(L,R)}, z_{w(L,R)}$ respectively. And their homotopy class in $\pi_1(T_{w(L,R)})$ or $\pi_1(\partial T_{w(L,R)})$ are denoted by $[x_{w(L,R)}]$, $[y_{w(L,R)}]$, $[z_{w(L,R)}]$ respectively.*

Proposition 2.9 *There exists two loops $x'_{w(L,R)}$ and $y'_{w(L,R)}$ embedded in $\partial T_{w(L,R)}$ satisfying the following conditions:*

(a) Two loops $x'_{w(L,R)}$ and $y'_{w(L,R)}$ intersect at one point in $\partial T_{w(L,R)}$;

(b) Two loops $x'_{w(L,R)}$ and $y'_{w(L,R)}$ are homotopic to the loops $x_{w(L,R)}$ and $y_{w(L,R)}$ in $\partial T_{w(L,R)}$ respectively.

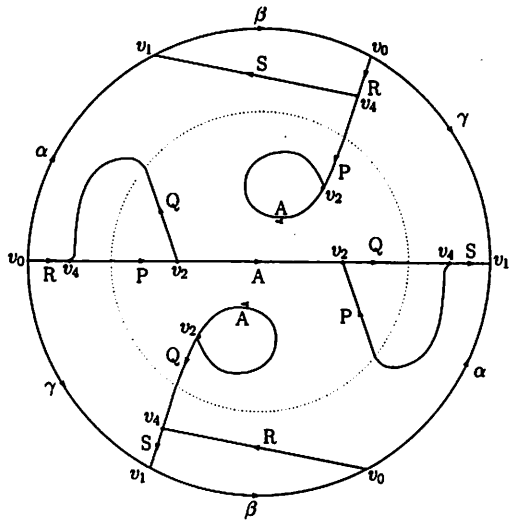

Figure 8: L-diagram

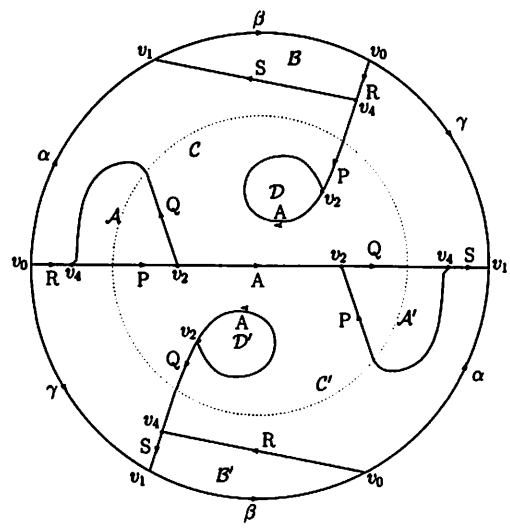

Figure 9: D_L

The homotopy class of three loops $x_{w(L,R)}$, $y_{w(L,R)}$, $z_{w(L,R)}$ in $\pi_1(\partial T_{w(L,R)})$ are satisfies the following condition:

(c) $[z_{w(L,R)}] = [y_{w(L,R)}][x_{w(L,R)}]^{-1}$.

Proof.

(a)(b) By Proposition 2.7, the neighborhood of the edge $\beta_{w(L,R)}$ in $\partial T_{w(L,R)}$ are (i) or (ii) shown in Figure 17. In the case (ii), there exists a face bounded by $\alpha\beta\alpha$. This is contradiction because the boundary of $E_{w(L,R)}$ is $\alpha\beta\gamma\alpha^{-1}\beta^{-1}\gamma^{-1}$. Thus, the neighborhoodof $\beta_{w(L,R)}$ is (i). Thereby, in $\partial T_{w(L,R)}$, two loops $\alpha_{w(L,R)}\beta_{w(L,R)}$ ($=: x_{w(L,R)}$) and $\beta_{w(L,R)}\gamma_{w(L,R)}$ ($=: y_{w(L,R)}$) are homotopic to the loops $x'_{w(L,R)}$ and $y'_{w(L,R)}$ respectively shown in $(*)$.

(c) We get the following relation in $\pi_1(\partial T_{w(L,R)})$:x

$$\begin{aligned}
[z_{w(L,R)}] &= [\gamma_{w(L,R)}\,\alpha^{-1}{}_{w(L,R)}] \\
&= [\gamma_{w(L,R)}\,\beta_{w(L,R)}\,\beta^{-1}{}_{w(L,R)}\,\alpha^{-1}{}_{w(L,R)}] \\
&= [\gamma_{w(L,R)}\,\beta_{w(L,R)}\,(\alpha_{w(L,R)}\,\beta_{w(L,R)})^{-1}].
\end{aligned}$$

By (a) and (b), two loops $\alpha_{w(L,R)}\,\beta_{w(L,R)}$ and $\gamma_{w(L,R)}\,\beta_{w(L,R)}$ are homotopic to the two loops which intersects at one point. Thus, we have,

$$[z_{w(L,R)}] = [\gamma_{w(L,R)}\,\beta_{w(L,R)}][\alpha_{w(L,R)}\,\beta_{w(L,R)}]^{-1}.$$

∎

Remark 2.10 *By Proposition 2.9 (a),(b), any loop embedded in $\partial T_{w(L,R)}$ is represented by the element $a[x_{w(L,R)}]+b[y_{w(L,R)}]$ in $\pi_1(\partial T_{w(L,R)})$ uniquely for some $a, b \in \mathbb{Z}$ because the fundamental group of $\partial T_{w(L,R)}$ is commutative.*

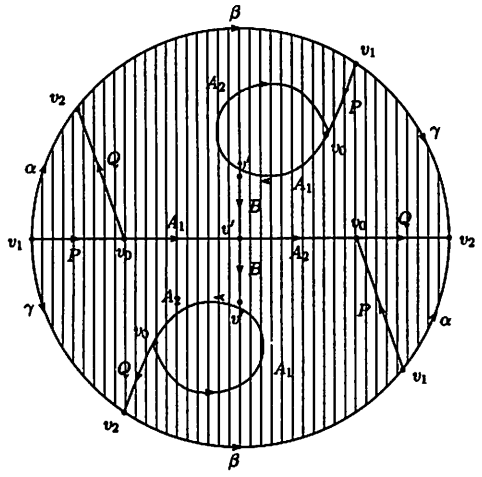

Figure 10: D_ϕ

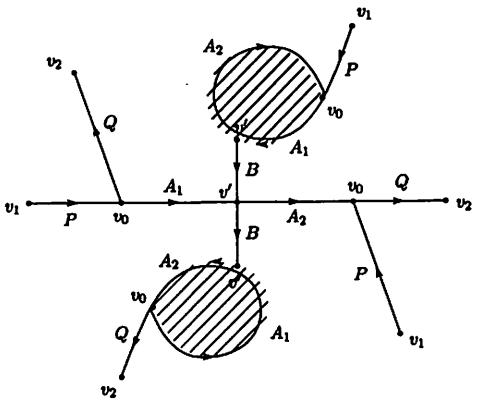

Figure 11: D'_ϕ

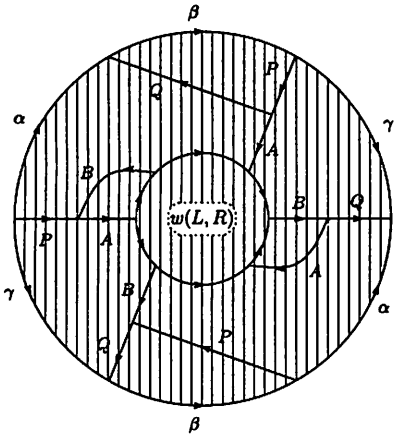

Figure 12: $D_{w(L,R)L}$

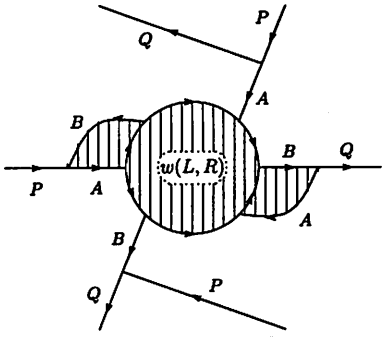

Figure 13: $D'_{w(L,R)L}$

Now we consider the manifold obtained from an X-ring, where $X = L, R, \bar{L}, \bar{R}$. Assume that an X-ring is embedded in S^2 ($\cong \partial B^3$).

Notation 2.11 *In S^2 in which an X-ring is embedded, the disc E', the annulus A and the disc $E := S^2 \setminus (E' \cup A))$ shown in Figure 18 are denoted by $E'_{\hat{X}}$, $A_{\hat{X}}$ and $E_{\hat{X}}$ respectively.*

Let $f_{\hat{X}}$ be an identification map on S^2 ($\cong \partial B^3$) induced from the identifying the directed labeled edges of X-ring.

Proposition 2.12 *Both $E'_{\hat{X}} / f_{\hat{X}}$ and $E_{\hat{X}} / f_{\hat{X}}$ are homeomorphic to $S^1 \times S^1$. And θ'-curve shown in Figure 19 and θ-curve shown in Figure 16 are embedded in $E'_{\hat{X}} / f_{\hat{X}}$ and $E_{\hat{X}} / f_{\hat{X}}$ respectively such that both $(E'_{\hat{X}}/f_{\hat{X}}) \setminus (\alpha' \cup \beta' \cup \gamma')$ and $(E_{\hat{X}}/f_{\hat{X}}) \setminus (\alpha \cup \beta \cup \gamma)$ are homeomorphic to an open 2-disc.*

Proof. The proof is similar to Proposition 2.7. ∎

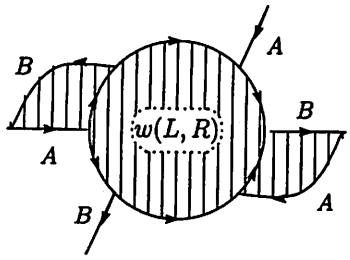

Figure 14: $D''_{w(L,R)L}$

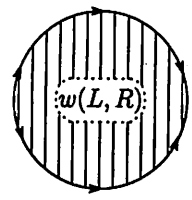

Figure 15: $D'''_{w(L,R)L}$

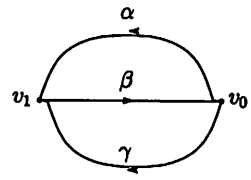

Figure 16: θ-curve

Notation 2.13 *We denote the arcs α, β, γ of θ-curve shown in Figure 16 embedded in $E_{\hat{X}} / f_{\hat{X}}$ by $\alpha_{\hat{X}}$, $\beta_{\hat{X}}$, $\gamma_{\hat{X}}$, respectively. Also,we denote the arcs α', β', γ' of θ shown in Figure 19 embedded in $E'_{\hat{X}} / f_{\hat{X}}$ by $\alpha'_{\hat{X}}$, $\beta'_{\hat{X}}$, $\gamma'_{\hat{X}}$ respectively.*

3 Lemmas

In this section, we prepare three lemmas.

Definition 3.1 *We define as follows:*

$$U_0 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$U_L := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}; \qquad U_R := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix};$$

$$U_{\bar{L}} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \qquad U_{\bar{R}} := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

where $U_{\bar{L}} = U_L^{-1}$ and $U_{\bar{R}} = U_R^{-1}$.

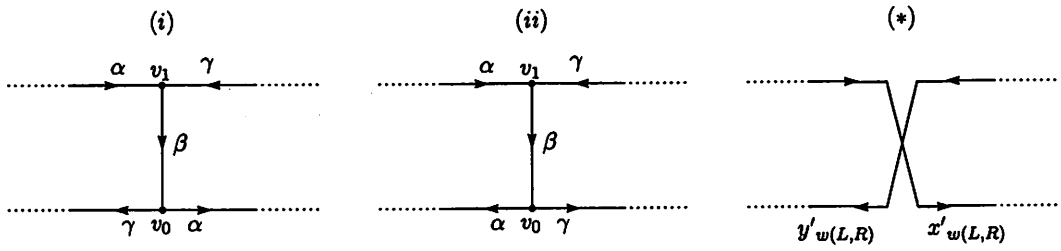

Figure 17: Neighborhood of the edge $\beta_{w(L,R)}$ embedded in $\partial T_{w(L,R)}$

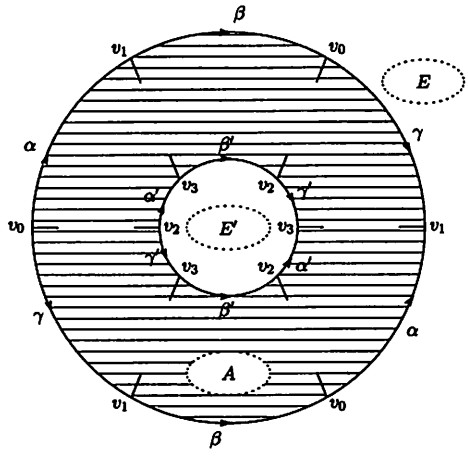

Figure 18: E', A, E

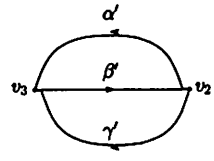

Figure 19: θ'

For integers $k_1, k_2, k_3, \ldots, k_{n-2}, k_{n-1}, k_n$,

$$d(k_1, k_2, k_3, \ldots, k_{n-2}, k_{n-1}, k_n) := \begin{pmatrix} k_1 & 1 & & & & & \\ -1 & k_2 & 1 & & & & \\ & -1 & k_3 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & k_{n-2} & 1 & \\ & & & & -1 & k_{n-1} & 1 \\ & & & & & -1 & k_n \end{pmatrix},$$

where the empty element is 0;

$D(k_1, k_2, k_3, \ldots, k_{n-2}, k_{n-1}, k_n)$

$:=$ *the determinant of the matrix* $d(k_1, k_2, k_3, \ldots, k_{n-2}, k_{n-1}, k_n)$;

$D_{i,j}(k_1, k_2, k_3, \ldots, k_{n-2}, k_{n-1}, k_n)$

$:=$ *the determinant of the matrix derived i-row and j-colimn of* $d(k_1, k_2, \ldots, k_{n-1}, k_n)$.

Lemma 3.2 *Let* $A[n] := \{a_1, a_2, \ldots, a_{n-1}, a_n\}$ *be a finite sequence of a natural number*

and the matrix $U(A[n])$ is defined as follows:

$$U(A[n]) := \begin{cases} U_0 U_L^{a_1} U_R^{a_2} U_L^{a_3} \cdots U_L^{a_{n-2}} U_R^{a_{n-1}} U_L^{a_n} & (n : odd) \\ U_0 U_L^{a_1} U_R^{a_2} U_L^{a_3} \cdots U_R^{a_{n-2}} U_L^{a_{n-1}} U_R^{a_n} & (n : even) \end{cases}$$
$$= \begin{pmatrix} X_\lambda & Y_\lambda \\ X_\mu & Y_\mu \end{pmatrix}.$$

Then, $U(A[n])$ satisfies the following conditions:

(a) The case n is even

$$X_\lambda = -D_{n+1,n+1}(a_1, \ldots, a_n, 1); \quad X_\mu = -D_{n+2,n+2}(0, a_1, \ldots, a_n, 1);$$
$$Y_\lambda = -D_{n,n+1}(a_1, \ldots, a_n, 1); \quad Y_\mu = -D_{n+1,n+2}(0, a_1, \ldots, a_n, 1); \tag{1}$$
$$X_\lambda Y_\mu - X_\mu Y_\lambda = -1; \tag{2}$$
$$-D(a_1, \ldots, a_n, 1)X_\mu + D(0, a_1, \ldots, a_n, 1)X_\lambda = -1;$$
$$-D(a_1, \ldots, a_n, 1)Y_\mu + D(0, a_1, \ldots, a_n, 1)Y_\lambda = -1. \tag{3}$$

(b) The case n is odd

$$X_\lambda = D_{n,n+1}(a_1, \ldots, a_n, 1); \quad X_\mu = D_{n+1,n+2}(0, a_1, \ldots, a_n, 1);$$
$$Y_\lambda = D_{n+1,n+1}(a_1, \ldots, a_n, 1); \quad Y_\mu = D_{n+2,n+2}(0, a_1, \ldots, a_n, 1); \tag{4}$$
$$X_\lambda Y_\mu - X_\mu Y_\lambda = -1; \tag{5}$$
$$-D(a_1, \ldots, a_n, 1)X_\mu + D(0, a_1, \ldots, a_n, 1)X_\lambda = -1;$$
$$-D(a_1, \ldots, a_n, 1)Y_\mu + D(0, a_1, \ldots, a_n, 1)Y_\lambda = -1. \tag{6}$$

Proof. We prove by mathematical induction on n. For convenience, we use the following notations:

$$D' := D(a_1, a_2, \ldots, a_{n-2}, a_{n-1}, 1); \qquad D := D(a_1, a_2, \ldots, a_{n-1}, a_n, 1);$$
$$D^{*\prime} := D(0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, 1); \qquad D^* := D(0, a_1, a_2, \ldots, a_{n-1}, a_n, 1);$$
$$D'_{i,j} := D_{i,j}(a_1, a_2, \ldots, a_{n-2}, a_{n-1}, 1); \qquad D_{i,j} := D_{i,j}(a_1, a_2, \ldots, a_{n-1}, a_n, 1);$$
$$D^{*\prime}{}_{i,j} := D_{i,j}(0, a_1, a_2, \ldots, a_{n-1}, 1); \qquad D^*{}_{i,j} := D_{i,j}(0, a_1, a_2, \ldots, a_n, 1).$$

1. The case $n \geq 2$

   Suppose the case $n-1$ is satisfied. Then, we will show the case n.

   (a) The case n is even

   We assume that

$$\begin{aligned} U(A[n-1]) &= \begin{pmatrix} D'_{n-1,n} & D'_{n,n} \\ D^{*\prime}{}_{n,n+1} & D^{*\prime}{}_{n+1,n+1} \end{pmatrix} \\ &= \begin{pmatrix} X'_\lambda & Y'_\lambda \\ X'_\mu & Y'_\mu \end{pmatrix}; \\ X'_\lambda Y'_\mu - X'_\mu Y'_\lambda &= -1; \\ -D'X_\mu + D^{*\prime}X_\lambda &= -1; \\ -D'Y_\mu + D^{*\prime}Y_\lambda &= -1. \end{aligned}$$

Then, we will show

$$
\begin{aligned}
U(A[n]) &= \begin{pmatrix} -D_{n+1,n+1} & -D_{n,n+1} \\ -D^*{}_{n+2,n+2} & -D^*{}_{n+1,n+2} \end{pmatrix} \\
&= \begin{pmatrix} X_\lambda & Y_\lambda \\ X_\mu & Y_\mu \end{pmatrix}; \qquad (7) \\
X'_\lambda Y'_\mu - X'_\mu Y'_\lambda &= -1; \qquad (8) \\
-D' X_\mu + D^{*\prime} X_\lambda &= -1; \qquad (9) \\
-D' Y_\mu + D^{*\prime} Y_\lambda &= -1; \qquad (10)
\end{aligned}
$$

i. We show equation (7)
By definition of $U(A[n])$,

$$
\begin{aligned}
\begin{pmatrix} X_\lambda & Y_\lambda \\ X_\mu & Y_\mu \end{pmatrix} &= U(A[n-1])(U_R)^{a_n} \\
&= \begin{pmatrix} X'_\lambda & Y'_\lambda \\ X'_\mu & Y'_\mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_n & 1 \end{pmatrix} \\
&= \begin{pmatrix} X'_\lambda - a_n Y'_\lambda & Y'_\lambda \\ X'_\mu - a_n Y'_\mu & Y'_\mu \end{pmatrix}.
\end{aligned}
$$

By calculation of matrix and determinant, we have

$$
\begin{aligned}
&-D_{n+1,n+1} \\
&= -\begin{vmatrix} a_1 & 1 & & & & & \\ -1 & a_2 & 1 & & & & \\ & -1 & a_3 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & a_{n-2} & 1 & \\ & & & & -1 & a_{n-1} & 1 \\ & & & & & -1 & a_n \end{vmatrix} \\
&= -a_n \begin{vmatrix} a_1 & 1 & & & \\ -1 & a_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & a_{n-2} & 1 \\ & & & -1 & a_{n-1} \end{vmatrix} + \begin{vmatrix} a_1 & 1 & & & \\ -1 & a_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & a_{n-2} & 1 \\ & & & 0 & -1 \end{vmatrix}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
X_\lambda &= X'_\lambda - a_n Y'_\lambda \\
&= D'_{n-1,n} - a_n D'_{n,n} \\
&= -D_{n+1,n+1}.
\end{aligned}
$$

Similarly, we have $Y_\lambda = -D_{n,n+1}$, $X_\mu = -D^*{}_{n+2,n+2}$, $Y_\mu = -D^*{}_{n+1,n+2}$.

ii. We show equation (8)

$$\begin{aligned} X_\lambda Y_\mu - X_\mu Y_\lambda &= (X'_\lambda - a_n Y'_\lambda) Y'_\mu - (X'_\mu - a_n Y'_\mu) Y'_\lambda \\ &= X'_\lambda Y'_\mu - a_n Y'_\lambda Y'_\mu - X'_\mu Y'_\lambda + a_n Y'_\mu Y'_\lambda \\ &= X'_\lambda Y'_\mu - X'_\mu Y'_\lambda \\ &= -1. \end{aligned}$$

iii. We show equations (9) and (10).
Expanding the determinant D and D^* with respect to the $n+1$-th column and $n+2$-th column respectively, we get $D = D_{n+1,n+1} - D_{n,n+1}$ and $D^* = D_{n+2,n+2} - D_{n+1,n+2}$. Thus, we have

$$\begin{aligned} -DX_\mu + D^* X_\lambda &= -(D_{n+1,n+1} - D_{n,n+1}) D^*_{n+1,n+2} \\ &\qquad +(D_{n+2,n+2} - D_{n+1,n+2}) D_{n,n+1} \\ &= -D_{n+1,n+1} D^*_{n+1,n+2} + D_{n+2,n+2} D_{n,n+1} \\ &= -Y_\lambda X_\mu + Y_\lambda X_\mu \\ &= -1. \end{aligned}$$

$$\begin{aligned} -DY_\mu + D^* Y_\lambda &= -(D_{n+1,n+1} - D_{n,n+1}) D^*_{n+2,n+2} \\ &\qquad +(D^*_{n+2,n+2} - D^*_{n+1,n+2}) D_{n+1,n+1} \\ &= D_{n,n+1} D^*_{n+2,n+2} - D^*_{n+1,n+2} D_{n+1,n+1} \\ &= X_\lambda Y_\mu - X_\mu Y_\lambda \\ &= -1. \end{aligned}$$

(b) The case n is odd
By calculations of determinants, equations (4), (5) and (6) will be proven similarly to the case (a).

Thereby, the rest of proof is to show the case $n = 1$.

2. The case $n = 1$

We get

$$\begin{aligned} U(A[1]) &= U_0 U_L{}^{a_1} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a_1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & a_1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

On the other hand, we have

$$X_\lambda = D_{1,2}\,(a_1,\,1) = -1; \qquad X_\mu = D_{2,3}\,(0,\,a_1,\,1) = \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} = 0;$$

$$Y_\lambda = D_{2,2}\,(a_1,\,1) = a_1; \qquad Y_\mu = D_{3,3}\,(0,\,a_1,\,1) = \begin{vmatrix} 0 & 1 \\ -1 & a_1 \end{vmatrix} = 1;$$

$$X_\lambda Y_\mu - X_\mu Y_\lambda = (-1)\cdot 1 - 0\cdot a_1 = -1.$$

∎

Now we show another lemma. We use the following notation for the fractional expansion:

$$[\,k_1,\,k_2,\,\ldots\,,k_{n-1},\,k_n\,] := \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{\ddots + \cfrac{1}{k_n}}}}.$$

In the following lemma, the equation (a) and (b) is shown in [9].

Lemma 3.3 *For a finite sequence of integer* $\{\,k_1,\,k_2,\,\ldots\,,\,k_n\,\}$, *we have* (a) *and* (b).

(a) $D\,(\,k_1,\,k_2,\,\ldots\,,\,k_n\,) = k_1\,D\,(\,k_2,\,k_3,\,\ldots\,,\,k_n\,) + D\,(\,k_3,\,k_4,\,\ldots\,,\,k_n\,)$.

(b) $[\,k_1,\,k_2,\,\ldots\,,k_{n-1},\,k_n\,] = \dfrac{D\,(\,0,\,k_1,\,k_2,\,\ldots\,,\,k_n\,)}{D\,(\,k_1,\,k_2,\,\ldots\,,\,k_n\,)}$.

For a finite sequence of natural number $\{\,k_1,\,k_2,\,\ldots\,,\,k_n\,\}$, *we get* (c)-(g):

(c) $D\,(\,0,\,k_1,\,k_2,\,\ldots\,,\,k_n\,) > 0$, $D\,(\,k_1,\,k_2,\,\ldots\,,\,k_n\,) > 0$.

(d) $D\,(\,0,\,k_1,\,k_2,\,\ldots\,,\,k_n\,)$ *and* $D\,(\,k_1,\,k_2,\,\ldots\,,\,k_n\,)$ *are coprime.*

(e) $D\,(\,0,\,k_1,\,k_2,\,\ldots\,,\,k_n,\,-1\,) \le 0$, $D\,(\,k_1,\,k_2,\,\ldots\,,\,k_n,\,-1\,) \le 0$.

(f) $-D\,(\,0,\,k_1,\,k_2,\,\ldots\,,\,k_n,\,-1\,)$ *and* $-D\,(\,k_1,\,k_2,\,\ldots\,,\,k_n,\,-1\,)$ *are coprime.*

(g) $D_{n+1,n+1}\,(\,a_1,\,\ldots\,,\,a_n\,)$ *and* $-D_{n,n+1}\,(\,a_1,\,\ldots\,,\,a_n\,)$ *are coprime.*

(h) $-D^*_{n+2,n+2}\,(\,a_1,\,\ldots\,,\,a_n\,)$ *and* $D^*_{n+1,n+2}\,(\,a_1,\,\ldots\,,\,a_n\,)$ *are coprime.*

For a pair of coprime natural number $p, q\,(\,p > q\,)$, *we obtain* (h):

(i) We choose a sequence of natural numbers $\{\,k_1,\,k_2,\,\ldots\,,\,k_n\,\}$ *such that* $q/p = [\,k_1,\,k_2,\,\ldots\,,k_{n-1},\,k_n,\,1\,]$. *Then we have*

$$D_{n+1,n+1}\,(\,k_1,\,k_2,\,\ldots\,,\,k_n,\,1\,) - D_{n,n+1}\,(\,k_1,\,k_2,\,\ldots\,,\,k_n,\,1\,) = p$$

Proof.

(a) We expand $D(k_1, k_2, \dots, k_n)$ with respect to 1-column.

$$
\begin{aligned}
&D(k_1, k_2, \dots, k_n) \\
&= k_1 \begin{vmatrix} k_2 & 1 & & & \\ -1 & k_3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & k_{n-1} & 1 \\ & & & -1 & k_n \end{vmatrix} + \begin{vmatrix} 1 & 0 & & & \\ -1 & k_3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & k_{n-1} & 1 \\ & & & -1 & k_n \end{vmatrix} \\
&= k_1 D(k_2, k_3, \dots, k_n) + D(k_3, k_4, \dots, k_n) \\
&\qquad + \begin{vmatrix} 0 & 0 & 0 & s & 0 \\ -1 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & k_{n-1} & 1 \\ & & & -1 & k_n \end{vmatrix} \\
&= k_1 D(k_2, k_3, \dots, k_n) + D(k_3, k_4, \dots, k_n)
\end{aligned}
$$

(b) We prove by mathematical induction on n. If $n = 1$, then we get $\dfrac{D(0, k_1)}{D(k_1)} = \dfrac{1}{k_1} = [k_1]$. Assume the case $n-1$ is satisfied. Then, we have

$$
\begin{aligned}
\frac{D(0, k_1, k_2, \dots, k_n)}{D(k_1, k_2, \dots, k_n)} &= \frac{0\,D(k_1, k_2, \dots, k_n) + D(k_2, k_3, \dots, k_n)}{k_1 D(k_2, k_3, \dots, k_n) + D(k_3, k_4, \dots, k_n)} \\
&= \frac{1}{k_1 + \dfrac{D(k_3, k_4, \dots, k_n)}{D(k_2, k_3, \dots, k_n)}} \\
&= \frac{1}{k_1 + \dfrac{D(0, k_2, k_3, \dots, k_n)}{D(k_2, k_3, \dots, k_n)}} \\
&= [k_1, k_2, \dots, k_{n-1}, k_n].
\end{aligned}
$$

(c) We prove by mathematical induction on n. If $n = 1$, then we get $D(k_1) = k_1 > 0$. If $n = 2$, then we have $D(k_1, k_2) = k_1k_2 + 1 > 0$. Assume that the claim is satisfied for all $i < n$. Then we will show the case n.

We get $D(k_1, k_2, \dots, k_n) = k_1 D(k_2, k_3, \dots, k_n) + D(k_3, k_4, \dots, k_n)$. By the assumption, we have $D(k_2, k_3, \dots, k_n) > 0$ and $D(k_3, k_4, \dots, k_n) > 0$. Thus, we have $D(k_1, k_2, \dots, k_n) > 0$.

By using (a), we obtain $D(0, k_1, k_2, \dots, k_n) = D(k_2, k_3, \dots, k_n) > 0$.

(d) We prove by mathematical induction on n. If $n = 1$, then we get $D(k_1, -1) = -k_1 + 1 \leq 0$. If $n = 2$, then we have $D(k_1, k_2, -1) = -k_1(1 - k_2) - 1 \leq 0$. Assume that the claim is satisfies for all $i < n$. Then we will show the case n.

We get $D(k_1, k_2, \ldots, k_n, -1) = k_1 D(k_2, k_3, \ldots, k_n, -1) + D(k_3, k_4, \ldots, k_n, -1)$. By the assumption, we have $D(k_2, k_3, \ldots, k_n, -1) \leq 0$ and $D(k_3, k_4, \ldots, k_n, -1) \leq 0$. Thus, we have $D(k_1, k_2, \ldots, k_n, -1) \leq 0$.

By using (a), we obtain $D(0, k_1, k_2, \ldots, k_n, -1) = D(k_2, k_3, \ldots, k_n, -1) \leq 0$.

(e) We prove by mathematical induction on n.

1. The case $n = 1$
   Since $D(0, k_1) = 1$ and $D(k_1) = k_1$, $D(0, k_1)$ and $D(k_1)$ are coprime.
2. Assume that the case $n-1$ is satisfied. Then we will show the case n.
   We get

$$\begin{aligned} D(0, k_1, k_2, \ldots, k_n) &= 0\,D(k_1, k_2, \ldots, k_n) + D(k_2, k_3, \ldots, k_n) \\ &= D(k_2, k_3, \ldots, k_n); \\ D(k_1, k_2, \ldots, k_n) &= k_1 D(k_2, k_3, \ldots, k_n) + D(k_3, k_4, \ldots, k_n) \\ &= k_1 D(k_2, k_3, \ldots, k_n) + D(0, k_2, k_3, \ldots, k_n). \end{aligned}$$

   $D(k_2, k_3, \ldots, k_n)$ and $D(0, k_2, k_3, \ldots, k_n)$ are coprime by assumption. This completes the proof.

(f) The proof is similar to (e). We prove by mathematical induction on n.

1. The case $n = 1$
   Since $-D(0, k_1, -1) = 1$ and $-D(k_1, -1) = k_1 - 1$, $D(0, k_1, -1)$ and $D(k_1, -1)$ are coprime.
2. Assume that the case $n-1$ is satisfies. Then we get

$$-D(0, k_1, \ldots, k_n, -1) = -D(k_2, \ldots, k_n, -1).$$

$$\begin{aligned} &-D(k_1, \ldots, k_n, -1) \\ &\qquad = -k_1 D(k_2, \ldots, k_n, -1) - D(k_3, \ldots, k_n, -1) \\ &\qquad = k_1 \left(-D(k_2, \ldots, k_n, -1)\right) + \left(-D(0, k_2, \ldots, k_n, -1)\right). \end{aligned}$$

   $-D(k_2, k_3, \ldots, k_n - 1)$ and $-D(0, k_2, k_3, \ldots, k_n - 1)$ are coprime by assumption. This completes the proof.

(g) By definition of $D_{i,j}(k_1, \ldots, k_n)$, we get

$$D_{n+1,n+1}(a_1, \ldots, a_n, 1) = D(a_1, \ldots, a_n).$$

By calculation of determinant, we have

$$-D_{n,n+1}(a_1, \ldots, a_n, 1) = D(a_1, \ldots, a_{n-1}).$$

By Definition of $D(k_1, \ldots, k_n)$,

$$D(a_1, a_2, \ldots, a_n) = D(a_n, a_{n-1}, \ldots, a_1).$$

By (a), we have,

$$\begin{aligned} & D(a_{n-1}, a_{n-2}, \dots, a_1) \\ &= 0D(a_n, a_{n-1}, \dots, a_1) + D(a_{n-1}, a_{n-2}, \dots, a_1) \\ &= D(0, a_n, a_{n-1}, \dots, a_1). \end{aligned}$$

By (d), $D(a_n, a_{n-1}, \dots, a_1)$ and $D(0, a_n, a_{n-1}, \dots, a_1)$ are coprime. Thus, $D_{n+1,n+1}$ and $-D_{n,n+1}$ are coprime.

(h) The proof is similar to (g).

(i) From (b), we get $q/p = [k_1, k_2, \dots, k_{n-1}, k_n, 1] = \dfrac{D(0, k_1, k_2, \dots, k_n, 1)}{D(k_1, k_2, \dots, k_n, 1)}$. By (c) and (e), we have $D(k_1, k_2, \dots, k_n, 1) = p$. Expanding $D(k_1, k_2, \dots, k_n, 1)$ with respect to $n+1$ column, we get our claim. ∎

Lemma 3.4 *For $i = 1, 2$, let θ_i be a θ-curve shown in Figure 20 and T_i be a torus. Assume that there exists embeddings $\phi_i : \theta_i \hookrightarrow T_i$ such that $T_i \setminus \phi(\theta_i) \cong Int(D^2)$. Then, there exists a homeomorphism $\varphi_{12} : T_1 \to T_2$ such that $\varphi_{12}(\alpha_1) = \alpha_2$, $\varphi_{12}(\beta_1) = \beta_2$ and $\varphi_{12}(\gamma_1) = \gamma_2$.*

Proof. Let $\Phi_{12} : \theta_1 \to \theta_2$ be a map defined by identifying the oriented edges $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ respectively. Then, cutting off T_i along $\phi_i(\theta_i)$, we get the 2-disc D_i^2 shown in Figure 21. Thus, the map Φ_{12} is extended to the homeomorphism $\Phi_{12}^* : D_1 \to D_2$ using Alexander trick. Thus, there exists a homeomorphism $\varphi_{12} : T_1 \to T_2$. such that $\varphi_{12}(\alpha_1) = \alpha_2$, $\varphi_{12}(\beta_1) = \beta_2$ and $\varphi_{12}(\gamma_1) = \gamma_2$. ∎

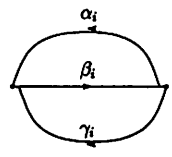

Figure 20: θ_i

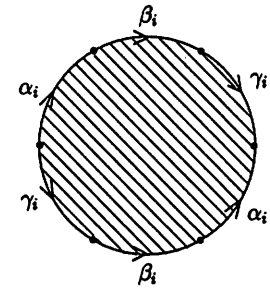

Figure 21: D_i^2

4 Relations between the elements of $\pi_1(T_\phi)$ and of $\pi_1(B^3/f_{\hat{X}})$

In this section, we consider the relations between the elements of the fundamental group of $B^3/f_{\hat{X}}$ and T_ϕ. Using those, we discuss about the meridian of $T_{w(L,R)}$ in section 5.

First we prepare the following proposition. Recall the definition of $D_{w(L,R)}$ and $A_{\hat{X}}$, see Notation 2.3 and Notation 2.11 respectively.

Proposition 4.1

1. $T_{w(L,R)} \setminus \left(\partial T_{w(L,R)} \cup (D_{w(L,R)} / f_{w(L,R)})\right)$ *is homeomorphic to an open 3-ball.*
2. $(B^3/f_{\hat{X}}) \setminus \left(\partial (B^3/f_{\hat{X}}) \cup (A_{\hat{X}} / f_{\hat{X}})\right)$ *is homeomorphic to an open 3-ball.*

Proof.

1. Recall that $T_{w(L,R)}$ is $B^3 / f_{w(L,R)}$. By definition, we have $E_{w(L,R)} \cup D_{w(L,R)} = \partial B^3$ and $E_{w(L,R)} / f_{w(L,R)} = \partial T_{w(L,R)}$. Thus, $T_{w(L,R)} \setminus \left(\partial T_{w(L,R)} \cup (D_{w(L,R)} / f_{w(L,R)})\right)$ is homeomorphic to $B^3 \setminus \partial B^3$.
2. The proof is similar to the case 1. ∎

At first, we consider the fundamental group of a solid torus T_ϕ. By Proposition 4.1, any loop embedded in T_ϕ is homotopic to some loop embedded in D_ϕ/f_ϕ. Thus, the fundamental group of T_ϕ is isomorphic to the fundamental group of D_ϕ/f_ϕ. Recall the notations $x_{w(L,R)}$ and $y_{w(L,R)}$, see Notation 2.8.

Proposition 4.2 *Let $[x_\phi]$, $[y_\phi]$ be homotopy class of two loops x_ϕ, y_ϕ in $\pi_1(T_\phi)$ respectively. Then, $\pi_1(T_\phi)$ is one generator free group and we get the following relations:*

$$\pi_1(T_\phi) =< P \mid - >; \qquad [x_\phi] = P; \qquad [y_\phi] = 0,$$

where the generator P is the representative element of the loop associated to edge P of ϕ-diagram shown in Figure 7.

Proof. We choose a maximal tree for $(Q \cup \beta)/f_\phi$ and fix a base point v_0. Then, we get

$$\begin{aligned}
\pi_1(T_\phi, v_0) &= \pi_1(D_\phi/f_\phi, v_0) \\
&= < \alpha, \gamma, P, A \mid P^{-1}\alpha, A\gamma^{-1}PAP^{-1}, A > \\
&= < \alpha, P, \gamma \mid P^{-1}\alpha, \gamma^{-1}PP^{-1} > \\
&= < P \mid - > .
\end{aligned}$$

Thus, we have the following relation in $\pi_1(T_\phi, v_0)$.

$$[x_\phi] = [\alpha_\phi\beta_\phi] = [P]; \qquad [y_\phi] = [\beta_\phi\gamma_\phi] = 0.$$

This relation depend on the choice neither of the base point nor of the maximal tree. ∎

We consider the fundamental group of $B^3/f_{\hat{X}}$, where $X = L, R, \bar{L}, \bar{R}$. Note that f_X is the identification map associated to the X-diagram and $f_{\hat{X}}$ is the identification map associated to the X-ring. Recall the notation $\alpha_{\hat{X}}$, $\beta_{\hat{X}}$, $\gamma_{\hat{X}}$, $\alpha'_{\hat{X}}$, $\beta'_{\hat{X}}$, $\gamma'_{\hat{X}}$, see Notation 2.13.

Proposition 4.3 *Let the elements $[\tilde{\alpha}_X]$, $[\tilde{\gamma}_X]$, $[\tilde{\alpha}'_X]$, $[\tilde{\gamma}'_X]$ in $\pi_1(B^3/f_{\hat{X}})$ be homotopy class of the loops $\alpha_{\hat{X}}\beta_{\hat{X}}$, $\beta_{\hat{X}}\gamma_{\hat{X}}$, $\alpha'_{\hat{X}}\beta'_{\hat{X}}$, $\beta'_{\hat{X}}\gamma'_{\hat{X}}$, embedded in $\partial(B^3/f_{\hat{X}})$ respectively. Then, $\pi_1(B^3/f_{\hat{X}})$ is commutative group and we get the following relations in $\pi_1(B^3/f_{\hat{X}})$:*

1. The case $X = L$

$$[\tilde{\alpha}_L] = [\tilde{\alpha}'_L]; \qquad [\tilde{\gamma}_L] = -[\tilde{\alpha}'_L] + [\tilde{\gamma}'_L].$$

2. The case $X = R$

$$[\tilde{\alpha}_R] = [\tilde{\alpha}'_R] - [\tilde{\gamma}'_R]; \qquad [\tilde{\gamma}_R] = [\tilde{\gamma}'_R].$$

3. The case $X = \bar{L}$

$$[\tilde{\alpha}_{\bar{L}}] = [\tilde{\alpha}'_{\bar{L}}]; \qquad [\tilde{\gamma}_{\bar{L}}] = [\tilde{\alpha}'_{\bar{L}}] + [\tilde{\gamma}'_{\bar{L}}].$$

4. The case $X = \bar{R}$

$$[\tilde{\alpha}_{\bar{R}}] = [\tilde{\alpha}'_{\bar{R}}] + [\tilde{\gamma}'_{\bar{R}}]; \qquad [\tilde{\gamma}_{\bar{R}}] = [\tilde{\gamma}'_{\bar{R}}].$$

Proof.

1. The case $X = L$

   By Proposition 4.1, any loop embedded in $B^3/f_{\hat{L}}$ is homotopic to some loop embedded in $A_{\hat{X}}/f_{\hat{L}}$. Thus, the fundamental group of $B^3/f_{\hat{L}}$ is isomorphic to the fundamental group of $A_{\hat{X}}/f_{\hat{L}}$.

   We choose a maximal tree of $A_{\hat{X}}/f_{\hat{L}}$ for $(Q \cup P \cup A \cup \beta')/f_{\hat{L}}$ and fix a base point v_3. Then, we get

$$\begin{aligned}
\pi_1(M_L, v_3) &= \pi_1(A_{\hat{L}}/f_{\hat{L}}, v_3) \\
&= \left\langle \begin{matrix} \alpha',\ \gamma',\ B \\ \alpha,\ \beta,\ \gamma \end{matrix} \;\middle|\; \begin{matrix} \alpha' B,\ B\alpha,\ \beta,\ \gamma B^{-1}\gamma'^{-1} \\ \alpha'\gamma'\alpha'^{-1}\gamma'^{-1},\ \alpha\beta\gamma\alpha^{-1}\beta^{-1}\gamma^{-1} \end{matrix} \right\rangle \\
&= \left\langle \begin{matrix} \alpha',\ \gamma',\ B \\ \alpha,\ \gamma \end{matrix} \;\middle|\; \begin{matrix} \alpha' B,\ B\alpha,\ \gamma B^{-1}\gamma'^{-1} \\ \alpha'\gamma'\alpha'^{-1}\gamma'^{-1},\ \alpha\gamma\alpha^{-1}\gamma^{-1} \end{matrix} \right\rangle \\
&= \left\langle \alpha',\ \gamma',\ \alpha,\ \gamma \;\middle|\; \begin{matrix} \alpha'^{-1}\alpha,\ \gamma\alpha'\gamma'^{-1} \\ \alpha'\gamma'\alpha'^{-1}\gamma'^{-1},\ \alpha\gamma\alpha^{-1}\gamma^{-1} \end{matrix} \right\rangle \\
&= \ll \alpha',\ \gamma',\ \alpha,\ \gamma \mid -\alpha' + \alpha,\ \gamma + \alpha' - \gamma' \gg,
\end{aligned}$$

   where $\ll\gg$ means omitting commutative relations. This relation depends on neither of the choice of the base point nor of the maximal tree.

2. The case $X = R$

   We choose maximal tree of $A_{\hat{R}}/f_{\hat{R}}$ for $(Q \cup P \cup A \cup \beta')/f_{\hat{R}}$ and we calculate $\pi_1(B^3/f_{\hat{R}})$ similar to the case 1.

3. The case $X = \bar{L}, \bar{R}$

   We choose maximal tree of $A_{\hat{X}}/f_{\hat{X}}$ for $(B \cup P \cup A \cup \beta)/f_{\hat{X}}$ and we calculate $\pi_1(B^3/f_{\hat{X}})$ similar to the case 1, where $X = \bar{L}, \bar{R}$ ∎

5 Meridian and longitude of $T_{w(L,R)}$

In this section, we discuss the meridian and longitude of $T_{w(L,R)}$, where *meridian* $m_{w(L,R)}$ of $T_{w(L,R)}$ is defined as a loop embedded in $\partial T_{w(L,R)}$ such that $[m_{w(L,R)}] = 0$ in $\pi_1(T_{w(L,R)})$ and *longitude* $l_{w(L,R)}$ of $T_{w(L,R)}$ is defined as a loop embedded in $\partial T_{w(L,R)}$ such that $[l_{w(L,R)}] \neq 0$ in $\pi_1(T_{w(L,R)})$ and $l_{w(L,R)}$ intersects $m_{w(L,R)}$ at one point in $\partial T_{w(L,R)}$.

Recall the loops $x_{w(L,R)}$, $y_{w(L,R)}$ embedded in $\partial T_{w(L,R)}$, see Notation 2.8. By Remark 2.10, any loop embedded in $\partial T_{w(L,R)}$ is represented by the element $a[x_{w(L,R)}] + b[y_{w(L,R)}]$ $(a, b \in \mathbb{Z})$ uniquely. In Proposition 5.2 and Proposition 5.5, we decide the coefficient a, b in the case $w(L,R) = \phi$ and $w(L,R) \neq \phi$ respectively.

First we consider the case $w(L,R) = \phi$.

Notation 5.1 *Two loops y_ϕ and x_ϕ^{-1} embedded in ∂T_ϕ are denoted by m and l respectively.*

Proposition 5.2 *Let $[m]$ and $[l]$ be the homotopy class of the loops m and l. Then, we get the following conditions:*

1. *The loop m is the meridian of T_ϕ, that is, $[m] = 0$ in $\pi_1(T_\phi)$;*
2. *The loop l is longitude of T_ϕ, that is, $[l] \neq 0$ in $\pi_1(T_\phi)$ and two loops l and m are homotopic to the two loops which intersect one point each other;*
3. *$[z_\phi] = [l] + [m]$ in $\pi_1(\partial T_\phi)$.*

Proof.

1. By Notation 5.1 and Proposition 4.2, $[m] = [y_\phi] = 0$ in $\pi_1(T_\phi)$.
2. By Notation 5.1 and Proposition 4.2, $[l] = [x_\phi] \neq 0$ in $\pi_1(T_\phi)$. From Proposition 2.9(a), (b), x_ϕ and y_ϕ are homotopic to two loops intersecting at one point in ∂T_ϕ.
3. By Proposition 2.9(c), we have $[z_\phi] = [y_\phi] - [x_\phi]$ in $\pi(\partial T_\phi)$. Thus, $[z_\phi] = [l] + [m]$ in $\pi_1(\partial T_\phi)$.

∎

By Notation 5.1 and Proposition 5.2, we obtain the following relation in $\pi_1(T_\phi)$.

$$\begin{aligned} {}^t\begin{pmatrix} [x_\phi] \\ [y_\phi] \end{pmatrix} &= {}^t\begin{pmatrix} [l] \\ [m] \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= {}^t\begin{pmatrix} [l] \\ [m] \end{pmatrix} U_0, \end{aligned} \tag{11}$$

where ${}^t(\quad)$ means transposition and U_0 is defined in Definition 3.1.

Now, we consider the meridian and longitude of $T_{w(L,R)}$ in the case the word $w(L,R) \neq \phi$. By Proposition 2.7, Proposition 2.12 and Lemma 3.4, there exists a homeomorphism $\varphi : \partial T_{w(L,R)} \to E'_{\hat{X}}/f_{\hat{X}} \subset \partial M_{\hat{X}}$, where $M_{\hat{X}}$ is $B^3/f_{\hat{X}}$. The manifold obtained by gluing $T_{w(L,R)}$ and $M_{\hat{X}}$ by φ is denoted by $T_{w(L,R)} \cup_\varphi M_{\hat{X}}$.

Lemma 5.3 *For any word $w(L,R)$ and $X \in \{L, R, \bar{L}, \bar{R}\}$, the manifold $T_{w(L,R)} \cup_\varphi M_{\hat{X}}$ is homeomorphic to $T_{w(L,R)X}$.*

Proof. Recall the notation of $E_{w(L,R)}$ and $E'_{\hat{X}}$, see Notation 2.3 and Notation 2.11. By definition of gluing map φ, torus $E_{w(L,R)}/f_{w(L,R)}$ ($\subset \partial T_{w(L,R)}$) and torus $E'_{\hat{X}}/f_{\hat{X}}$ ($\subset \partial M_{\hat{X}}$) are identified such that $\varphi(\alpha) = \alpha'$, $\varphi(\beta) = \beta'$, $\varphi(\gamma) = \gamma'$, see Figure 22. Thus, the manifold $T_{w(L,R)} \cup_\varphi M_{\hat{X}}$ is obtained from by $B^3/f_{w(L,R)X}$. ∎

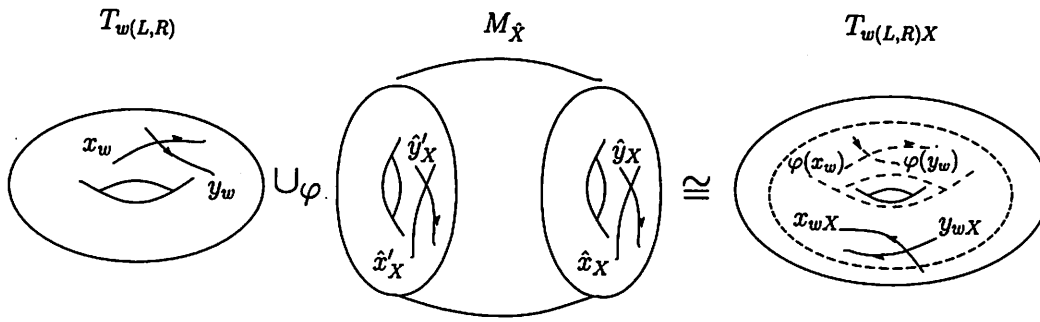

Figure 22: Gluing map φ

By gluing map φ, two loops $\varphi(x_{w(L,R)})$ and $\varphi(y_{w(L,R)})$ are embedded in $T_{w(L,R)X}$, see Figure 22. Thus, four loops $\varphi(x_{w(L,R)})$, $\varphi(y_{w(L,R)})$, $x_{w(L,R)X}$ and $y_{w(L,R)X}$ are embedded in $T_{w(L,R)X}$. Thus, there exists integers a, b, c, d such that we get the following equations in $\pi_1(T_{w(L,R)X})$:

$$\begin{aligned} [x_{w(L,R)X}] &= a[\varphi(x_{w(L,R)})] + b[\varphi(y_{w(L,R)})]; \\ [y_{w(L,R)X}] &= c[\varphi(x_{w(L,R)})] + d[\varphi(y_{w(L,R)})]. \end{aligned}$$

The following proposition tells us the integers a, b, c, d. For convenience, we denote $\varphi(x_{w(L,R)})$, $\varphi(y_{w(L,R)})$ by $x_{w(L,R)}$, $y_{w(L,R)}$ respectively.

Proposition 5.4 *For any word $w(L,R)$, homotopy class of the loops $x_{w(L,R)}$, $y_{w(L,R)}$, $x_{w(L,R)X}$, $y_{w(L,R)X}$ satisfy the following equation in $\pi_1(T_{w(L,R)X})$, where $X = L, R, \bar{L}, \bar{R}$:*

$$^t\begin{pmatrix} [x_{w(L,R)X}] \\ [y_{w(L,R)X}] \end{pmatrix} = {}^t\begin{pmatrix} [x_{w(L,R)}] \\ [y_{w(L,R)}] \end{pmatrix} U_X,$$

where U_X is defined in Definition 3.1.

Proof.

1. The case $X = L$

   Since $T_{w(L,R)}$ is solid torus, $\pi_1(T_{w(L,R)})$ is one generator free group. Thus, we suppose that

$$\pi_1(T_{w(L,R)}, v_0) =< G \mid - >; \qquad [x_{w(L,R)}] = mG; \qquad [y_{w(L,R)}] = lG,$$

   where v_0 is the vertex v_0 of $w(L,R)$ - diagram. We have

$$\begin{aligned} \pi_1(T_{w(L,R)} \cap M_{\hat{L}}, v_0) &= < \alpha',\ \gamma' \mid \alpha'\gamma'\alpha'^{-1}\gamma'^{-1} > \\ &= \ll \alpha',\ \gamma' \mid - \gg, \end{aligned}$$

   where $\ll\gg$ means omitting commutative relations. By Proposition 4.3,

$$\pi_1(M_{\hat{L}}, v_2) =\ll \alpha',\ \gamma',\ \alpha,\ \gamma \mid -\alpha' + \alpha,\ \gamma + \alpha' - \gamma' \gg,$$

   where v_2 is vertex v_2 of L - ring, see Figure 3. By gluing map φ, the vertex v_2 of $M_{\hat{L}}$ is identified with v_0 of $T_{w(L,R)}$. According to the theorem of Seifert-van Kampen, we get

$$\begin{aligned} &\pi_1(T_{w(L,R)L}, v_2) \\ &= \ll G,\ \alpha',\ \gamma',\ \alpha,\ \gamma \mid -\alpha' + \alpha,\ \gamma + \alpha' - \gamma',\ \alpha' = mG,\ \gamma' = lG \gg \\ &= \ll G,\ \alpha,\ \gamma \mid -mG + \alpha,\ \gamma + mG - lG \gg \\ &= \ll G \mid - \gg, \end{aligned}$$

   and

$$[\tilde{\alpha}_L] = mG = [\tilde{\alpha}'_L]; \qquad [\tilde{\gamma}_L] = -mG + lG = -[\tilde{\alpha}'_L] + [\tilde{\gamma}'_L].$$

   Thereby, we have the following relations:

$$\begin{aligned} [x_{w(L,R)X}] &= [\tilde{\alpha}_L] = mG = [\tilde{\alpha}'_L] = [x_{w(L,R)}]; \\ [y_{w(L,R)X}] &= [\tilde{\gamma}_L] = -mG + lG = -[\tilde{\alpha}'_L] + [\tilde{\gamma}'_L] - [x_{w(L,R)}] + [y_{w(L,R)}]. \end{aligned}$$

   Thus, we obtain the following relation in $\pi_1(T_{w(L,R)L})$:

$$\begin{aligned} {}^t\begin{pmatrix} [x_{w(L,R)L}] \\ [y_{w(L,R)L}] \end{pmatrix} &= {}^t\begin{pmatrix} [x_{w(L,R)}] \\ [y_{w(L,R)}] \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= {}^t\begin{pmatrix} [x_{w(L,R)}] \\ [y_{w(L,R)}] \end{pmatrix} U_L, \end{aligned}$$

   where U_L is defined in Definition 3.1.

2. The case $X = R, \bar{L}, \bar{R}$

   Calculating $\pi_1(T_{w(L,R)X}, v_0)$ similar to the case 1, we have

$$ {}^t\begin{pmatrix} [x_{w(L,R)X}] \\ [y_{w(L,R)X}] \end{pmatrix} = {}^t\begin{pmatrix} [x_{w(L,R)}] \\ [y_{w(L,R)}] \end{pmatrix} U_X, $$

   where $X = R, \bar{L}, \bar{R}$.

∎

Recall that two loops m and l are embedded in ∂T_ϕ. By Lemma 5.3, there exists two loops $m_{w(L,R)}$ and $l_{w(L,R)}$ embedded in $\partial T_{w(L,R)}$ such that

$$ \begin{aligned} [m_{w(L,R)}] &= [m] && \text{in } \pi_1(T_{w(L,R)}); \\ [l_{w(L,R)}] &= [l] && \text{in } \pi_1(T_{w(L,R)}). \end{aligned} $$

By Remark 2.10, there exists integer a, b, c, d such that

$$ \begin{aligned} [m_{w(L,R)}] &= a\,[x_{w(L,R)}] + b\,[y_{w(L,R)}] && \text{in } \pi_1(\partial T_{w(L,R)}); \\ [l_{w(L,R)}] &= c\,[x_{w(L,R)}] + d\,[y_{w(L,R)}] && \text{in } \pi_1(\partial T_{w(L,R)}). \end{aligned} $$

Proposition 5.6 tells us the coefficient a, b, c, d and $m_{w(L,R)}$ and $l_{w(L,R)}$ are the meridian and longitude of $T_{w(L,R)}$.

First we show the following proposition. Recall the notation $D_{i,j}$, $D^*_{i,j}$, D and D^* in Lemma 3.2.

Proposition 5.5 *For a finite sequence of a natural number $A[n] := \{a_1, a_2, \ldots, a_{n-1}, a_n\}$, we define a word on the letter $\{L, R\}$ as follow:*

$$ w(A[n]) := \begin{cases} L^{a_1} R^{a_2} L^{a_3} \cdots L^{a_{n-2}} R^{a_{n-1}} L^{a_n} & (n : odd) \\ L^{a_1} R^{a_2} L^{a_3} \cdots R^{a_{n-2}} L^{a_{n-1}} R^{a_n} & (n : even) \end{cases} . $$

Then, we get the following conditions:

(a) The case n is odd

1. $[m] = D_{n+1,n+1}\,[x_{w(A[n])}] - D_{n,n+1}\,[y_{w(A[n])}]$ *in* $\pi_1(T_{w(A[n])})$;
2. $[l] = -D^*_{n+2,n+2}\,[x_{w(A[n])}] + D^*_{n+1,n+2}\,[y_{w(A[n])}]$ *in* $\pi_1(T_{w(A[n])})$.

(b) The case n is even

1. $[m] = -D_{n,n+1}\,[x_{w(A[n])}] + D_{n+1,n+1}\,[y_{w(A[n])}]$ *in* $\pi_1(T_{w(A[n])})$;
2. $[l] = D^*_{n+1,n+2}\,[x_{w(A[n])}] - D^*_{n+2,n+2}\,[y_{w(A[n])}]$ *in* $\pi_1(T_{w(A[n])})$.

Proof.

(a) The case n is odd

By Proposition 5.4, we get

$$
\begin{aligned}
{}^t\begin{pmatrix} [x_{w(A[n])}] \\ [y_{w(A[n])}] \end{pmatrix} &= {}^t\begin{pmatrix} [x_\phi] \\ [y_\phi] \end{pmatrix} U_L{}^{a_1} U_R{}^{a_2} U_L{}^{a_3} \cdots U_L{}^{a_{n-2}} U_R{}^{a_{n-1}} U_L{}^{a_n} \\
&= {}^t\begin{pmatrix} [l] \\ [m] \end{pmatrix} U_0 U_L{}^{a_1} U_R{}^{a_2} U_L{}^{a_3} \cdots U_L{}^{a_{n-2}} U_R{}^{a_{n-1}} U_L{}^{a_n}.
\end{aligned}
$$

By Lemma 3.2, we have

$$
\begin{aligned}
&{}^t\begin{pmatrix} [l] \\ [m] \end{pmatrix} U_0 U_L{}^{a_1} U_R{}^{a_2} U_L{}^{a_3} \cdots U_L{}^{a_{n-2}} U_R{}^{a_{n-1}} U_L{}^{a_n} \\
&\quad = {}^t\begin{pmatrix} [l] \\ [m] \end{pmatrix} \begin{pmatrix} D_{n,n+1} & D_{n+1,n+1} \\ D^*{}_{n+1,n+2} & D^*{}_{n+2,n+2} \end{pmatrix}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{cases} [x_{w(A[n])}] &= D_{n,n+1}[l] + D^*_{n+1,n+2}[m] \\ [y_{w(A[n])}] &= D_{n+1,n+1}[l] + D^*_{n+2,n+2}[m] \end{cases}
$$

$$
\leftrightarrow \begin{cases} [m] &= D_{n+1,n+1}[x_{w(A[n])}] - D_{n,n+1}[y_{w(A[n])}] \\ [l] &= -D^*_{n+2,n+2}[x_{w(A[n])}] + D^*_{n+1,n+2}[y_{w(A[n])}] \end{cases}.
$$

(b) The case n is even

By Proposition 5.4, we can prove similarly to the case (a). ∎

Proposition 5.6 *The meridian $m_{w(A[n])}$ and the longitude $l_{w(A[n])}$ of solid torus $T_{w(A[n])}$ are represented by as follows:*

(a) The case n is odd

1. $[m_{w(A[n])}] = D_{n+1,n+1}[x_{w(A[n])}] - D_{n,n+1}[y_{w(A[n])}]$ *in* $\pi_1(\partial T_{w(A[n])})$;
2. $[l_{w(A[n])}] = -D^*_{n+2,n+2}[x_{w(A[n])}] + D^*_{n+1,n+2}[y_{w(A[n])}]$ *in* $\pi_1(\partial T_{w(A[n])})$.

(b) The case n is even

1. $[m_{w(A[n])}] = -D_{n,n+1}[x_{w(A[n])}] + D_{n+1,n+1}[y_{w(A[n])}]$ *in* $\pi_1(\partial T_{w(A[n])})$;
2. $[l_{w(A[n])}] = D^*_{n+1,n+2}[x_{w(A[n])}] - D^*_{n+2,n+2}[y_{w(A[n])}]$ *in* $\pi_1(\partial T_{w(A[n])})$.

Proof.

(a) The case n is odd

By Proposition 5.5, $[m_{(A[n])}] = [m]$ in $\pi_1(T_{w(A[n])})$. By Proposition 5.2, $[m] = 0$ in $\pi_1(T_{w(A[n])})$. Thus, $[m_{(A[n])}] = 0$ in $\pi_1(T_{w(A[n])})$. Similarly, $[l_{(A[n])}] = [l] \neq 0$ in $\pi_1(T_{w(A[n])})$.

Thus, the rest of proof is to show the following equations:

i. $D_{n+1,n+1}$ and $-D_{n,n+1}$ are coprime.

ii. $-D^*_{n+2,n+2}$ and $D^*_{n+1,n+2}$ are coprime.

iii. $D_{n+1,n+1}D^*_{n+1,n+2} - D_{n,n+1}(-D^*_{n+2,n+2}) = 1$ or -1

These are shown in Lemma 3.3(g),(h) and the equation (5) in Lemma 3.2 respectively.

(b) The case n is even

Similar to the case (a).

∎

6 Lens space obtained by gluing $T_{w(L,R)}$ and $T_{w'(L,R)}$

In this section, we define the gluing of two solid tori $T_{w(L,R)}$ and $T_{w'(L,R)}$. Then, we show the following fact in Theorem 6.1: for any pair of coprime natural number $p, q\,(p > q)$, we can choose two words $w(L,R)$ and $w'(L,R)$ such that $L(p,q)$ is homeomorphic to the manifold obtained by gluing of two solid tori $T_{w(L,R)}$ and $T_{w'(L,R)}$.

First we see the gluing of two solid tori $T_{w(L,R)}$ and $T_{w'(L,R)}$. For convenience, $w(L,R)$ and $w'(L,R)$ are denoted by w and w'. By Proposition 2.7 and Lemma 3.4, there exists a homeomorphism $\varphi : \partial T_{w(L,R)} \to \partial T_{w'(L,R)}$. The manifold obtained from gluing T_w and $T_{w'}$ by φ is denoted by $T_w \cup_\varphi T_{w'}$.

Theorem 6.1 *For any pair of coprime natural number $p, q\,(p > q)$, we choose a sequence of natural numbers $\{a_1, a_2, \ldots, a_{n-1}, a_n\}$ such that the fractional expansion of q/p is $[a_1, a_2, \ldots, a_{n-1}, a_n, -1]$ and define a word $v(q/p)$ as follows:*

$$v(q/p) \;:=\; \begin{cases} L^{a_1}R^{a_2}L^{a_3}\cdots L^{a_{n-2}}R^{a_{n-1}}L^{a_n} & (n : odd) \\ L^{a_1}R^{a_2}L^{a_3}\cdots R^{a_{n-2}}L^{a_{n-1}}R^{a_n} & (n : even) \end{cases}.$$

Then, we get $L(p,q) \cong T_L \cup_\varphi T_{v(q/p)}$. In particular, $S^3 \cong T_L \cup_\varphi T_{LL}$ and $S^2 \times S^1 \cong T_L \cup_\varphi T_L$.

Proof.

1. We show $L(p,q) \cong T_L \cup_\varphi T_{v(q/p)}$

   We use the same notations $D_{i,j}$ and $D^*_{i,j}$ in Lemma 3.2. For convenience, we denote a word $v(q/p)$ by v. By Proposition 5.6, we get the following relation about the meridian m_L of the solid torus T_L:

$$[m_L] = [x_L] + [y_L]; \qquad \text{in } \pi_1(\partial T_L). \quad (12)$$

(a) The case of n is odd

By Proposition 5.6, we have the following relations about the meridian m_v of the solid torus T_v:

$$[m_v] = D_{n+1,n+1}[x_v] - D_{n,n+1}[y_v]; \qquad in\ \pi_1(\partial T_v). \tag{13}$$

By Proposition 5.6, we have the following relations about the longitude l_v of the solid torus T_v:

$$[l_v] = -D^*_{n+2,n+2}[x_v] + D^*_{n+1,n+2}[y_v]; \qquad in\ \pi_1(\partial T_v). \tag{14}$$

According to definition of φ, we obtain

$$\varphi(x_L) = x_v; \qquad \varphi(y_L) = y_v. \tag{15}$$

Let $\varphi^\# : \pi_1(\partial T_L) \to \pi_1(\partial T_v)$ be a homomorphism induced by homeomorphism φ. Using the equations (12) and (15), we have

$$\begin{aligned}
[\varphi(m_L)] &= \varphi^\#([m_L]) \\
&= \varphi^\#([x_L] + [y_L]) \\
&= \varphi^\#([x_L]) + \varphi^\#([y_L]) \\
&= [\varphi(x_L)] + [\varphi(y_L)] \\
&= [x_v] + [y_v].
\end{aligned} \tag{16}$$

By equations (13), (14) and (16), we get the following equation in $\pi_1(\partial T_v)$:

$$[\varphi(m_L)] = (D^*_{n+2,n+2} + D^*_{n+1,n+2})[m_v] + (D_{n+1,n+1} + D_{n,n+1})[l_v].$$

We have the following equations:

$$\begin{aligned}
D_{n+1,n+1} + D_{n,n+1} &= D(a_1, a_2, \dots, a_{n-1}, a_n - 2, 1) \\
&= D(a_1, a_2, \dots, a_{n-1}, a_n, -1); \\
D^*_{n+2,n+2} + D^*_{n+1,n+2} &= D(0, a_1, a_2, \dots, a_{n-1}, a_n - 2, 1) \\
&= D(0, a_1, a_2, \dots, a_{n-1}, a_n, -1).
\end{aligned}$$

Thus, we have

$$\frac{D^*_{n+2,n+2} + D^*_{n+1,n+2}}{D_{n,n+1} + D_{n+1,n+1}} = \frac{D(0, a_1, a_2, \dots, a_{n-1}, a_n, -1)}{D(a_1, a_2, \dots, a_{n-1}, a_n, -1)}.$$

By Lemma 3.3(b) and the definition of $\{a_1, a_2, \dots, a_{n-1}, a_n\}$, we get

$$\frac{D(0, a_1, a_2, \dots, a_{n-1}, a_n, -1)}{D(a_1, a_2, \dots, a_{n-1}, a_n, -1)} = [a_1, a_2, \dots, a_{n-1}, a_n, -1] = \frac{q}{p}.$$

By Lemma 3.3(e), $D(0, a_1, \dots, a_n, -1) \le 0$ and $D(a_1, \dots, a_n, -1) \le 0$. From Lemma 3.3(f), $-D(0, a_1, \dots, a_n, -1)$ and $-D(a_1, \dots, a_n, -1)$ are coprime. So, we have $D(0, a_1, \dots, a_n, -1) = -q$, $D(a_1, \dots, a_n, -1) = -p$. Thus, we obtain $(D^*_{n+2,n+2} + D^*_{n+1,n+2}, D_{n,n+1} + D_{n+1,n+1}) = (-q, -p)$. Thereby, the manifold $T_L \cup_\varphi T_v$ is homeomorphic to the lens space $L(p, q)$.

(b) The case n is even

By Proposition 5.5, we have the following equation in $\pi_1(\partial T_v)$:

$$[\varphi(m_L)] = (D^*_{n+2,n+2} + D^*_{n+1,n+2})[m_v] + (D_{n+1,n+1} + D_{n,n+1})[l_v].$$

Thereby, the manifold $T_L \cup_\varphi T_v$ is homeomorphic to the lens space $L(p,q)$.

2. We show $S^3 \cong T_L \cup_\varphi T_{LL}$

By Proposition 5.6, we get

$$[\varphi(m_L)] = [m_{LL}] + [l_{LL}].$$

3. We show $S^2 \times S^1 \cong T_L \cup_\varphi T_L$

By Proposition 5.6, we get

$$[\varphi(m_L)] = [m_L]$$

∎

7 Type of $T_{w(L,R)}$

For a pair of coprime natural number $p, q\,(p > q)$, we will define a word $w(q/p)$ on the letters $\{L, R\}$ and we call $T_{w(q/p)}$ *q/p type solid torus.* In section 8, we will see that q/p type solid torus corresponds to (p,q) type singular fiber.

First we show the following proposition. Recall that the loop $z_{w(L,R)}$ is defined as the loop $\gamma_{w(L,R)}\alpha_{w(L,R)}{}^{-1}$ embedded in $\partial T_{w(L,R)}$. We denote the homotopy class in $\pi_1(\partial T_{w(L,R)})$ of the loop $z_{w(L,R)}$ by $[z_{w(L,R)}]$.

Proposition 7.1 *For any pair of coprime natural number $p, q\,(p > q)$, we choose a sequence of natural numbers $\{a_1, a_2, \ldots, a_{n-1}, a_n\}$ such that the fractional expansion of q/p is $[a_1, a_2, \ldots, a_{n-1}, a_n, 1]$ and define the word $w(q/p)$ as follow:*

$$w(q/p) := \begin{cases} L^{a_1} R^{a_2} L^{a_3} \cdots L^{a_{n-2}} R^{a_{n-1}} L^{a_n} & (n : odd) \\ L^{a_1} R^{a_2} L^{a_3} \cdots R^{a_{n-2}} L^{a_{n-1}} R^{a_n} & (n : even) \end{cases}.$$

Then, we get the following relation in $\pi_1(\partial T_{w(q/p)})$:

$$[z_{w(q/p)}] = p\,[l_{w(q/p)}] + q\,[m_{w(q/p)}],$$

where $l_{w(q/p)}$ and $m_{w(q/p)}$ are defined in Proposition 5.6.

Proof. Recall the notation $D_{i,j}, D^*_{i,j}, D, D^*$ in Lemma 3.2. By Proposition 2.9(c), we have the following relation in $\pi_1(\partial T_{w(q/p)})$:

$$[z_{w(q/p)}] \;=\; [y_{w(q/p)}] - [x_{w(q/p)}].$$

(a) The case n is odd

By equations (6) in Lemma 3.2, we have $DX_\mu - D^*X_\lambda = 1$ and $-DY_\mu + D^*Y_\lambda = -1$. For convenience, we denote $w(q/p)$ by w in calculation. According to Proposition 5.6, we get the following relation in $\pi_1(\partial T_w)$:

$$\begin{aligned}
[z_w] &= [y_w] - [x_w] \\
&= (DX_\mu - D^*X_\lambda)[y_w] + (-DY_\mu + D^*Y_\lambda)[x_w] \\
&= D(-Y_\mu[x_w] + X_\mu[y_w]) + D^*(Y_\lambda[x_w] - X_\lambda[y_w]) \\
&= D(-D^*_{n+2,n+2}[x_w] + -D^*_{n+1,n+2}[y_w]) \\
&\qquad\qquad + D^*(D_{n+1,n+1}[x_w] + -D_{n,n+1}[y_w]) \\
&= D[l_w] + D^*[m_w].
\end{aligned}$$

(b) The case n is even

By Lemma 3.2 and Proposition 5.6, we get the following equation similar to the case (a).

$$[z_w] = D[l_w] + D^*[m_w].$$

According to the Lemma 3.3(b) and the definition of $\{a_1, a_2, \ldots, a_{n-1}, a_n\}$, we get

$$\begin{aligned}
\frac{D^*}{D} &= \frac{D(0, a_1, a_2, \ldots, a_{n-1}, a_n, 1)}{D(a_1, a_2, \ldots, a_{n-1}, a_n, 1)} \\
&= [a_1, a_2, \ldots, a_{n-1}, a_n, 1] \\
&= q/p\,.
\end{aligned}$$

We also have $D(0, a_1, a_2, \ldots, a_{n-1}, a_n, 1)$ and $D(a_1, a_2, \ldots, a_{n-1}, a_n, 1)$ are positive and coprime by the Lemma 3.3(c),(d). Thus we get $D(0, a_1, a_2, \ldots, a_{n-1}, a_n, 1) = q$ and $D(a_1, a_2, \ldots, a_{n-1}, a_n, 1) = p$. ∎

Notation 7.2 *For a word $w(q/p)$ defined in Proposition 7.1, $T_{w(q/p)}$ is called q/p type solid torus and denoted by $T(q/p)$.*

Now, we define b *type solid torus* for a non-zero integer b.

Definition 7.3 *For an integer $b\,(\neq 0)$, we define the word $w(b)$ as follows:*

$$LR^{b-1}\bar{L}\ (b > 0); \qquad\qquad L\bar{R}^{-b+1}\bar{L}\ (b < 0).$$

For the solid torus $T_{w(b)}$, two loops $x_{w(b)} := \alpha_{w(b)}\beta_{w(b)}$ and $y_{w(b)} := \beta_{w(b)}\gamma_{w(b)}$ are embedded in $\partial T_{w(b)}$. Recall that any loop embedded in $\partial T_{w(b)}$ is represented uniquely by $c_0[x_{w(b)}] + c_1[y_{w(b)}]$ $(c_0, c_1 \in \mathbb{Z})$, see Remark 2.10.

Definition 7.4 *We define two loops $m_{w(b)}$ and $l_{w(b)}$ embedded in $\partial T_{w(b)}$ represented by the element in $\pi_1(\partial T_{w(b)})$:*

$$\begin{aligned} [m_{w(b)}] &:= (1-b)[x_{w(b)}] + b[y_{w(b)}]; \\ [l_{w(b)}] &:= (b-2)[x_{w(b)}] + (1-b)[y_{w(b)}]. \end{aligned}$$

Proposition 7.5 *1. $[m_{w(b)}]$ is meridian of $T_{w(b)}$, that is, $[m_{w(b)}] = 0$ in $\pi_1(T_{w(b)})$;*

2. $[l_{w(b)}]$ is longitude of $T_{w(b)}$, that is, $[l_{w(b)}] \neq 0$ in $\pi_1(T_{w(b)})$ and $m_{w(b)}$ and $l_{w(b)}$ intersects at one point.

Proof.

(a) The case $b > 0$

By Proposition 5.4, we get

$$\begin{aligned} {}^t\begin{pmatrix} [x_{w(b)}] \\ [y_{w(b)}] \end{pmatrix} &= {}^t\begin{pmatrix} [x_\phi] \\ [y_\phi] \end{pmatrix} U_L\, {U_R}^{b-1}\, U_{\bar{L}} \\ &= {}^t\begin{pmatrix} [l] \\ [m] \end{pmatrix} U_0\, U_L\, {U_R}^{b-1}\, U_{\bar{L}} \\ &= {}^t\begin{pmatrix} [l] \\ [m] \end{pmatrix} \begin{pmatrix} -b & -b+1 & 1 \\ -b+1 & -b+2 & 1 \end{pmatrix}, \end{aligned}$$

where U_0, U_L, U_R, $U_{\bar{L}}$ are defined in Lemma 3.2. Thus, we have the following relation in $\pi_1(T_{w(b)})$.

$$\begin{cases} [x_{w(b)}] = (-b)[l] + (-b+1)[m] \\ [y_{w(b)}] = (-b+1)[l] + (-b+2)[m] \end{cases}$$

$$\leftrightarrow \begin{cases} [m] = (1-b)[x_{w(b)}] + b[y_{w(b)}] \\ [l] = (b-2)[x_{w(b)}] + (1-b)[y_{w(b)}] \end{cases}.$$

And we have $[m] = 0$ in $\pi_1(T_{w(b)})$ and $[l] \neq 0$ in $\pi_1(T_{w(b)})$. Thus, two loops $m_{w(b)}$ and $l_{w(b)}$ embedded in $\partial T_{w(b)}$ are meridian and longitude respectively.

(b) The case $b < 0$

The proof is similar to the case (a). ∎

Recall the loop $z_{w(L,R)}$ embedded in $\partial T_{w(b)}$ is defined as the loop $\gamma_{w(L,R)}{\alpha^{-1}}_{w(L,R)}$.

Proposition 7.6 *For any integer $b \neq 0$, we get the following relation in $\pi_1(\partial T_{w(b)})$:*

$$[z_{w(b)}] = [l_{w(b)}] + [m_{w(b)}].$$

Proof. By Proposition 2.9, we get the following relation in $\pi_1(\partial T_{w(b)})$:

$$\begin{aligned}[z_{w(b)}] &= [y_{w(b)}] - [x_{w(b)}] \\ &= (-b+1)[l_{w(b)}] + (-b+2)[m_{w(b)}] \\ &\qquad - \big((-b)[l_{w(b)}] + (-b+1)[m_{w(b)}]\big) \\ &= [l_{w(b)}] + [m_{w(b)}].\end{aligned}$$

∎

Definition 7.7 *For a word $w(b)$ defined in Definition 7.3, $T_{w(b)}$ is called b type solid torus and denoted by T(b).*

8 Seifert manifold

In this section, we define the identification maps f_{G_n} and f_* on ∂B^3 such that B^3/f_{G_n} and B^3/f_* are homeomorphic to $\left(S^2 - \left(\coprod_{i=1}^{n+2} Int(D_i^2)\right)\right) \times S^1$ and $(S^1 \times S^1 - D^2) \times S^1$ respectively. And we show that any Seifert manifold whose base space is orientable can be obtained by gluing $T(q_i/p_i), T(b), B^3/f_{G_n}$ and B^3/f_*.

We use the notation $H_i\,(i \in \mathbb{N})$ for the diagram shown in Figure 23. Assume that H_i is embedded in ∂B^3. Let f_{H_i} be an identification map on ∂B^3 associated with the diagram H_i, that is, the edges which have same oriented label are identified and the identification of vertices and face are induced by the identification of edges.

First we consider the manifold B^3/f_{H_i}, denoted by M_{H_i}.

Proposition 8.1 *M_{H_i} is homeomorphic to $\left(S^2 - \left(\coprod_{i=1}^{3} Int(D_i^2)\right)\right) \times S^1$.*

Proof. This proof is similar to [10]. Without loss of generality, we assume the following conditions:

1. $B^3 = \{(x,y,z) \in \mathbb{R}^3\,;\, x^2+y^2+z^2 \le 1\}$;
2. $\partial B^3 \cap \{z=0\}$ is the bold line drown shown in Figure 24.
3. The projection of the edge X_1 embedded in $\partial B^3 \cap \{z>0\}$ to the surface $z=0$ and of the edge X_1 embedded in $\partial B^3 \cap \{z<0\}$ to the surface $z=0$ are congruent. Similarly, the edges X_2, X_3, Y_j and Z_j are satisfies the same condition $(j=1,2,3)$.

Suppose the flow generated by the vector field $\partial/\partial z$ on B^3. A point a in region A shown in Figure 25 is mapped by the identification map f_{H_i} to a_1 in the region A_1. And it is moved by the flow $\partial/\partial z$ and arrives at a_2 in A_2. After that, it is mapped to a_3 in A_3 by f_{H_i} and it turns back by a flow to the same point a in A. Also a point in region ∂A turns back to the same point in ∂A by f_{H_i} and the flow $\partial/\partial z$. Any point in M_{H_i} turns back in the disc $A \cup B \cup C \cup D \cup E$ shown in Figure 25. Thus, the base space of M_{H_i} is H_i'/f_{H_i}, where H_i' is shown in Figure 26. Since H_i'/f_{H_i} is homeomorphic to $S^2 - \left(\coprod_{i=1}^{3} Int(D_i^2)\right)$,

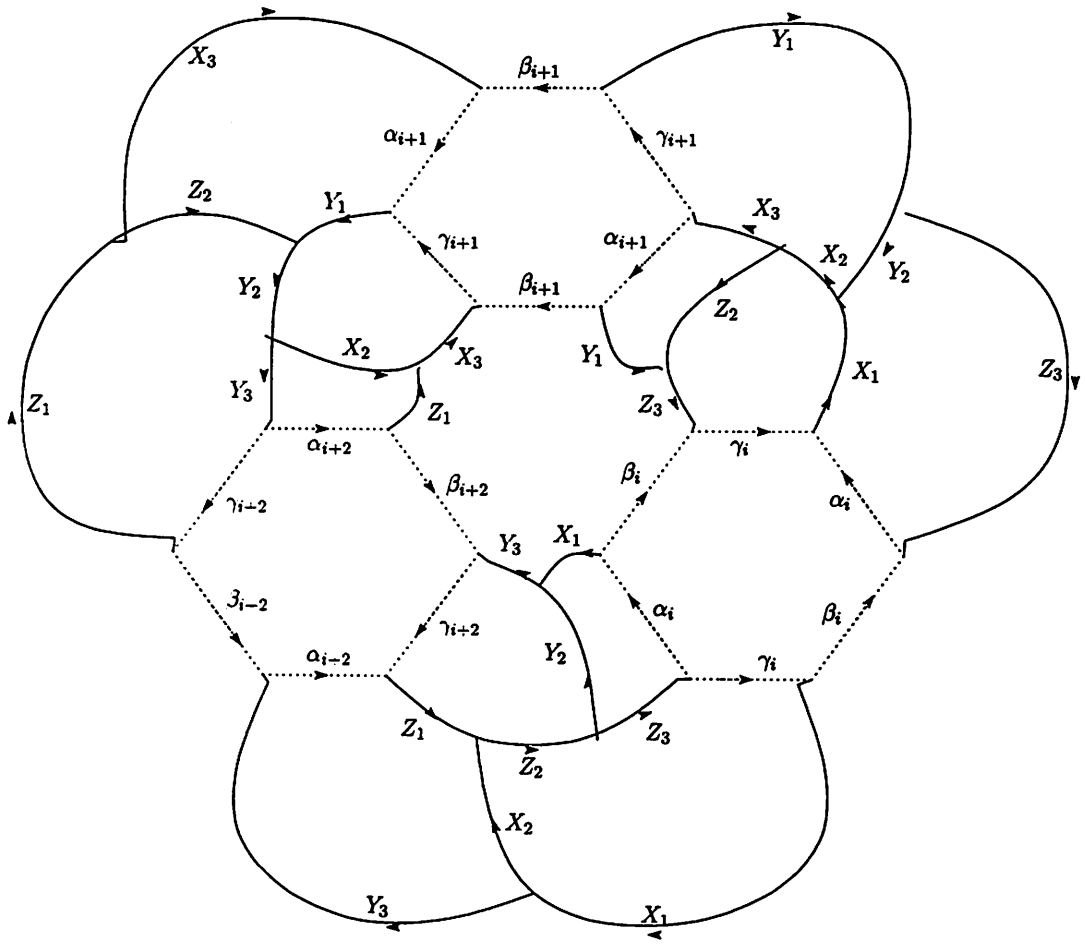

Figure 23: Diagram H_i

the manifold M_{H_i} is homeomorphic to $\left(S^2 - \left(\coprod_{i=1}^{3} Int(D_i^2)\right)\right) \times S^1$. ∎

By the proof of Proposition 8.1, a fiber structure of M_{H_i} by the vector field $\partial/\partial z$. The loop $\{(x,y,z) \mid x = y = 0,\ -1 \leq z \leq 1\}/f_{H_i}\ (\cong S^1)$ is a fiber of M_{H_i}. It is isotopic to $\gamma_k \alpha_k^{-1}$ $(k = i, i+1, i+2)$.

Now we define the diagram G_n inductively by using H_i.

Definition 8.2 *The diagram G_1 is defined as H_1. Suppose the diagram G_{n-1} is defined. Then, G_n is defined the following two steps.*

1. *Identify the circle $\alpha_{n+1}\,\beta_{n+1}\,\gamma_{n+1}\,\alpha_{n+1}^{-1}\,\beta_{n+1}^{-1}\,\gamma_{n+1}^{-1}$ of G_{n-1} and the circle $\alpha_n\,\beta_n\,\gamma_n\,\alpha_n^{-1}\,\beta_n^{-1}\,\gamma_n^{-1}$ of H_n such that the directed edges α_{n+1}, β_{n+1}, γ_{n+1} and α_n, β_n, γ_n are identified respectively.*

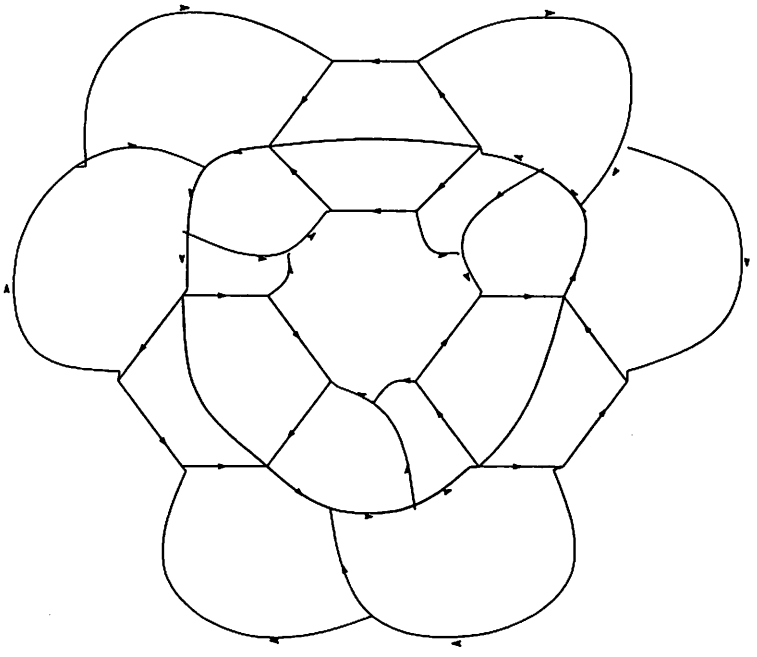

Figure 24: $\partial B^3 \cap \{z = 0\}$

2. Delete the edge $\alpha_{n+1}\,\beta_{n+1}\,\gamma_{n+1}$ of G_{n-1} and $\alpha_n\,\beta_n\,\gamma_n$ of H_n.

Assume that G_n is embedded in ∂B^3. Let f_{G_n} be an identification map on ∂B^3. induced from the identifying the directed labeled edges of G_n. We consider the manifold B^3/f_{G_n}, denoted by M_{G_n}.

Proposition 8.3 *The manifold M_{G_n} is homeomorphic to $\left(S^2 - \left(\coprod_{i=1}^{n+2} Int\,(D_i^2)\right)\right) \times S^1$.*

Proof. The proof is similar to Proposition 8.1. Suppose the flow $\partial/\partial z$. Contracting a flow to a point, we get the disc G'_n which is homeomorphic to $S^2 - \left(\coprod_{i=1}^{n+2} Int\,(D_i^2)\right)$. Thus, M_{G_n} is homeomorphic to $\left(S^2 - \left(\coprod_{i=1}^{n+2} Int\,(D_i^2)\right)\right) \times S^1$. ∎

By the proof of Proposition 8.1, a fiber structure of M_{G_n} is $\partial/\partial z$. Thus, a loop $\gamma_i\alpha_i^{-1}$ is a fiber of M_{G_n}, where $1 \le i \le n+2$.

Now, we consider the manifold $(S^1 \times S^1 - D^2) \times S^1$. The diagram shown in Figure 27 is called $*i$-diagram and denoted by $*i$. Assume that $*i$-diagram is embedded in ∂B^3. Let f_{*i} be an identification map on ∂B^3 associated with the $*$-diagram. We consider the manifold B^3/f_{*i}, denoted by M_{*i}.

Proposition 8.4 *M_{*i} is homeomorphic to $(S^1 \times S^1 - D^2) \times S^1$.*

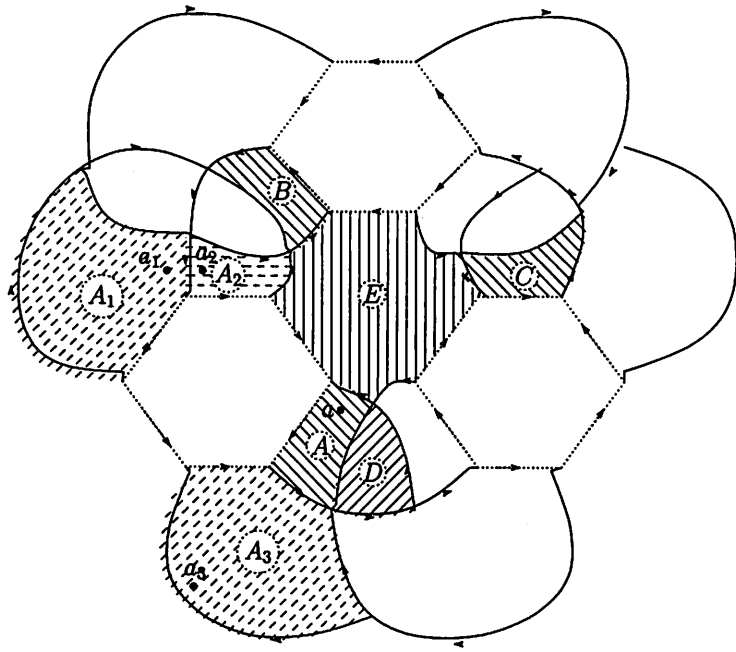

Figure 25: Region A,B,C,D and E

Proof. The proof is similar to Proposition 8.1. Embeds $*i$-diagram in the boundary of the unit ball and suppose the flow $\partial/\partial z$. Contracting a flow to a point, we get the surface $*_i'$ which is homeomorphic to $S^1 \times S^1 - D^2$. Thus, M_{*i} is homeomorphic to $(S^1 \times S^1 - D^2) \times S^1$. ∎

Now, we consider a Seifert fibered space $S\,(F_g,\, b;\, (\alpha_1, \beta_1), \ldots, (\alpha_{n+1}, \beta_{n+1}))$, where F_g is orientable closed surface with g genus. The boundary of the manifold $M_{G_{n+g}}$ is $\coprod_{i=1}^{n+g+2} D_i/f_{G_{n+g}}$, where D_i is a disc shown in Figure 21. Thus, $\partial M_{G_{n+g}} = \coprod_{i=1}^{n+g+2} T_i$, where $T_i := D_i/f_{G_{n+g}} \cong S^1 \times S^1$. And θ_i-curve is embedded in T_i such that $T_i \setminus (\alpha_i \cup \beta_i \cup \gamma_i)$ is homeomorphic to an open disc. So, there exists a homeomorphism $\varphi_i : \partial T(q_i/p_i) \to T_i$ $(1 \le i \le n+1)$ and $\varphi_{n+2} : \partial T(b) \to T_{n+2}$ and $\varphi_i : \partial M_{*i} \to T_i$ $(n+3 \le i \le n+g+2)$ by Lemma 3.4.

We use the notation $M_{G_{n+g}} \cup T(q_i/p_i) \cup T(b) \cup M_{*i}$ for the manifold obtained by gluing $M_{G_{n+g}}$ and $\coprod_{i=1}^{n+1} T(q_i/p_i)$ and $T(b)$ and $\coprod_{i=n+3}^{n+g+2} M_{*i}$ by φ_i.

Theorem 8.5 *The manifold $M_{G_{n+g}} \cup T(q_i/p_i) \cup T(b) \cup M_{*i}$ is homeomorphic to the Seifert manifold $S\,(F_g,\, b;\, (p_1, r_1), \ldots, (p_{n+1}, r_{n+1}))$, where r_i is decided such that $q_i/p_i = [a_1,\, a_2,\, \ldots, a_{k-1},\, a_k,\, 1]$ and*

$$r_i = \begin{cases} -D_{k,k+1}(a_1,\, a_2,\, \ldots,\, a_{k-1},\, a_k,\, 1) & (k : odd) \\ D_{k+1,k+1}(a_1,\, a_2,\, \ldots,\, a_{k-1},\, a_k\, 1) & (k : even) \end{cases},$$

where $D_{i,j}(a_1,\, a_2,\, \ldots,\, a_{n-1},\, a_n)$ in Definition 3.1

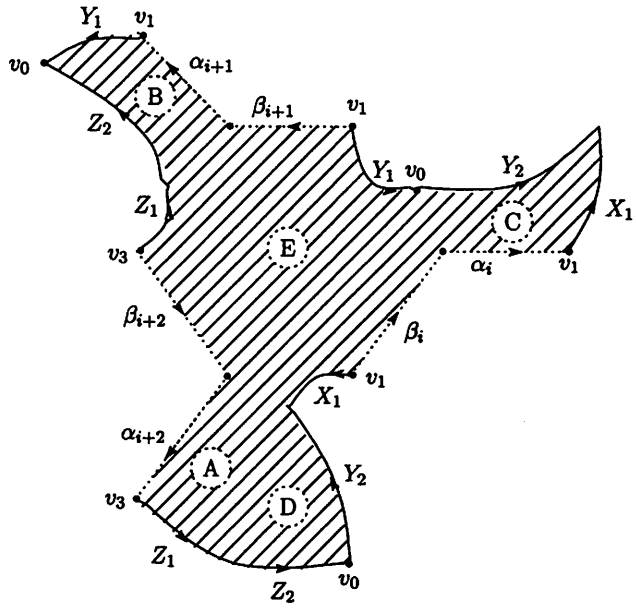

Figure 26: Base space H_i'

Proof.

1. The case $1 \leq i \leq n+1$

   For convenience, we denote $w(q_i/p_i)$ by w.

   (a) The case k is odd of q_i/p_i.

   By Proposition 5.6, the meridian of $T(q_i/p_i)$ satisfies the following conditions in $\pi_1(\partial T(q_i/p_i))$.

$$\begin{aligned} [m_w] &= D_{n+1,n+1}[x_w] - D_{n,n+1}[y_w]; \\ [l_w] &= -D^*_{n+2,n+2}[x_w] + D^*_{n+1,n+2}[y_w]. \end{aligned}$$

   By definition of φ_i, we get $\varphi_i^{-1}(\gamma_i\alpha_i^{-1}) = \gamma\alpha^{-1}$. Since $\gamma_i\alpha_i^{-1}$ is a fiber, we regard $\gamma\alpha^{-1}$ as a fiber of $T(q_i/p_i)$. Let $\varphi_i^{\#} : \pi_1(\partial T(q_i/p_i)) \to \pi_1(T_i)$ be a homomorphism induced by homeomorphism φ_i. By Proposition 7.1, we get the following equation in $\pi_1(\partial T(q_i/p_i))$.

$$\begin{aligned} [\varphi_i^{-1}(\gamma_i\alpha_i^{-1})] &= \varphi_i^{\#^{-1}}([\gamma_i\alpha_i^{-1}]) \\ &= [\gamma\alpha^{-1}] \\ &= [z_w] \\ &= p_i[l_w] + q_i[m_w]. \end{aligned}$$

   This means that the core of $T(q_i/p_i)$ is (p_i, q_i) type singular fiber. And we

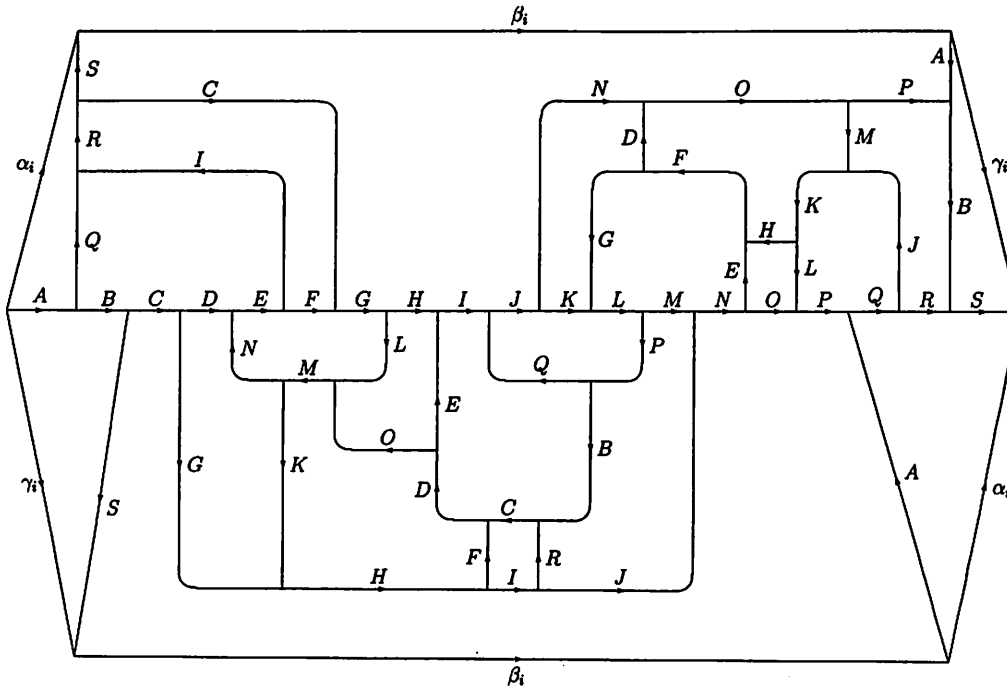

Figure 27: Diagram $*i$

have the following equation *in* $\pi_1(T_i)$:

$$\begin{aligned}
[\varphi_i(m_w)] &= \varphi_i^{\#}([m_w]) \\
&= \varphi_i^{\#}(D_{k+1,k+1}[x_w] - D_{k,k+1}[y_w]) \\
&= D_{k+1,k+1}\,\varphi_i^{\#}([x_w]) - D_{k,k+1}\,\varphi_i^{\#}([y_w]) \\
&= D_{k+1,k+1}[\alpha_i\beta_i] - D_{k,k+1}[\beta_i\gamma_i] \\
&= D_{k+1,k+1}[\alpha_i\beta_i] - D_{k,k+1}[\beta_i\alpha_i\alpha_i^{-1}\gamma_i] \\
&= D_{k+1,k+1}[\alpha_i\beta_i] - D_{k,k+1}([\beta_i\alpha_i] + [\gamma_i\alpha_i^{-1}]) \\
&= (D_{k+1,k+1} - D_{k,k+1})[\alpha_i\beta_i] - D_{k,k+1}[\gamma_i\alpha_i^{-1}]
\end{aligned}$$

Two loops $\alpha_i\beta_i$ and $\gamma_i\alpha_i^{-1}$ are embedded in T_i such that $\alpha_i\beta_i = T_i \cap H_i'$ and $\alpha_i\beta_i$ intersects $\gamma_i\alpha_i^{-1}$ at a point. By Lemma 3.3(i), we get $D_{k+1,k+1} - D_{k,k+1} = p_i$. So the fiber type is $(p_i, -D_{k,k+1})$.

(b) The case k is even

The proof is similar to the case (a). By Proposition 5.6, we have the following conditions in $\pi_1(q_i/p_i)$.

$$[\varphi_i^{-1}(\gamma_i\alpha_i^{-1})] = p_i[l_w] + q_i[m_w].$$

This means that the core of $T(q_i/p_i)$ is (p_i, q_i) type singular fiber. Also, we get

the following conditions in $\pi_1(T_i)$.

$$\begin{aligned}
[\varphi_i(m_w)] &= \varphi_i^{\#}(-D_{k,k+1}[x_w] + D_{k+1,k+1}[y_w]) \\
&= -D_{k,k+1}\varphi_i^{\#}([x_w]) + D_{k+1,k+1}\varphi_i^{\#}([y_w]) \\
&= (D_{k+1,k+1} - D_{k,k+1})[\alpha_i\beta_i] + D_{k+1,k+1}[\gamma_i\alpha_i^{-1}] \\
&= p_i[\alpha_i\beta_i] + D_{k+1,k+1}[\gamma_i\alpha_i^{-1}].
\end{aligned}$$

Thus, the fiber type is $(p_i, D_{k+1,k+1})$.

2. The case $i = n+2$

The proof is similar to the case 1. We get the following relation in $\pi_1(\partial T(b))$.

$$\begin{aligned}
[m_{w(b)}] &= (1-b)[x_{w(b)}] + b[y_{w(b)}]; \\
[l_{w(b)}] &= (b-2)[x_{w(b)}] + (1-b)[y_{w(b)}].
\end{aligned}$$

Since $\varphi_{n+2}^{-1}(\gamma_{n+2}\alpha_{n+2}^{-1}) = \gamma\alpha^{-1}$, the loop $\gamma\alpha^{-1}$ is a fiber of $T(b)$. By Proposition 7.5, we get the following equation in $\pi_1(\partial T(b))$.

$$\begin{aligned}
[\varphi_{n+2}^{-1}(\gamma_{n+2}\alpha_{n+2}^{-1})] &= \varphi_{n+2}^{\#\,-1}([\gamma_{n+2}\alpha_{n+2}^{-1}]) \\
&= [\gamma\alpha^{-1}] \\
&= [z_{w(b)}] \\
&= [l_{w(b)}] + [m_{w(b)}].
\end{aligned}$$

By Proposition 7.5, we get the following equation *in* $\pi_1(T_{n+2})$:

$$\begin{aligned}
[\varphi_{n+2}(m_{w(b)})] &= \varphi_{n+2}^{\#}([m_{w(b)}]) \\
&= \varphi_{n+2}^{\#}((1-b)[x_{w(b)}] + b[y_{w(b)}]) \\
&= (1-b)\varphi_{n+2}^{\#}([x_{w(b)}]) + b\varphi_{n+2}^{\#}([y_{w(b)}]) \\
&= (1-b)[\alpha_{n+2}\beta_{n+2}] + b[\beta_{n+2}\gamma_{n+2}] \\
&= (1-b)[\alpha_{n+2}\beta_{n+2}] + b[\beta_{n+2}\alpha_{n+2}\alpha_{n+2}^{-1}\gamma_{n+2}] \\
&= (1-b)[\alpha_{n+2}\beta_{n+2}] + b([\beta_{n+2}\alpha_{n+2}] + [\alpha_{n+2}^{-1}\gamma_{n+2}]) \\
&= [\alpha_{n+2}\beta_{n+2}] + b[\gamma_{n+2}\alpha_{n+2}^{-1}].
\end{aligned}$$

This means that the core of $T(b)$ is the fiber corresponding to the obstruction class b.

3. The case $n+3 \le i \le n+g+2$

By the definition of φ_i and Proposition 8.4, the base space of M_{*i} is $S^1 \times S^1 - D^2$. Thus, the base space of $M_{G_{n+g}} \cup T(q_i/p_i) \cup T(b) \cup M_{*i}$ has g genus.

∎

9 Singular triangulation of Seifert manifold

Definition 9.1 *For $i = 1, 2, \ldots, n+1$, let (p_i, q_i) be pairs of coprime natural numbers such that $p_i > q_i$ and b an integer such that $b \neq 0$ and g be a natural number. Then, the diagram $G_{n+g}(*_1, *_2, \ldots, *_g, b, q_1/p_1, q_2/p_2, \ldots, q_{n+1}/p_{n+1})$ is defined the following six steps.*

1. *Identify the circle $\alpha_i \beta_i \gamma_i \alpha_i^{-1} \beta_i^{-1} \gamma_i^{-1}$ of the G_{n+g} and the circle $\alpha \beta \gamma \alpha^{-1} \beta^{-1} \gamma^{-1}$ of $w(p_i/q_i)$ - diagram such that the directed edges α_i, β_i, γ_i and α, β, γ are identified respectively, where $i = 1, 2, \ldots, n+1$*

2. *Delete the edge α_i, β_i, γ_i of G_{n+g} and α, β, γ of an $w(p_i/q_i)$ - diagram, where $i = 1, 2, \ldots, n+1$*

3. *Identify the circle $\alpha_{n+2} \beta_{n+2} \gamma_{n+2} \alpha_{n+2}^{-1} \beta_{n+2}^{-1} \gamma_{n+2}^{-1}$ of the G_{n+g} and the circle $\alpha \beta \gamma \alpha^{-1} \beta^{-1} \gamma^{-1}$ of $w(b)$ - diagram such that the directed edges α_{n+2}, β_{n+2}, γ_{n+2} and α, β, γ are identified respectively.*

4. *Delete the edge α_{n+2}, β_{n+2}, $\gamma_{n+2} i$ of G_{n+g} and α, β, γ of an $w(b)$ - diagram.*

5. *Identify the circle $\alpha_i \beta_i \gamma_i \alpha_i^{-1} \beta_i^{-1} \gamma_i^{-1}$ of the G_{n+g} and the circle $\alpha_i \beta_i \gamma_i \alpha_i^{-1} \beta_i^{-1} \gamma_i^{-1}$ of $*i$ - diagram such that the directed edges α_i, β_i, γ_i and α_i, β_i, γ_i are identified respectively, where $i = n+2, n+3, \ldots, n+g+2$.*

6. *Delete the edge α_i, β_i, γ_i of G_{n+g} and α_i, β_i, γ_i of an $*i$ - diagram,*

where two words $w(p_i/q_i)$ and $w(b)$ are defined in Proposition 7.1 and Definition 7.3.

Remark 9.2 *The diagram $G_{n+g}(*_1, *_2, \ldots, *_g, b, q_1/p_1, q_2/p_2, \ldots, q_{n+1}/p_{n+1})$ is DS-diagram, where DS-diagram is defined in [4]. The DS-diagram of lens space is shown in [11] and [12].*

Assume that $G := G_{n+g}(*_1, *_2, \ldots, *_g, b, q_1/p_1, q_2/p_2, \ldots, q_{n+1}/p_{n+1})$ is embedded in ∂B^3. Let f_G be an identification map induced from the identifying the directed labeled edges of G. We consider B^3/f_G, which is denoted by $M\left(G_{n+g}(g_*, b, q_1/p_1, q_2/p_2, \ldots, q_{n+1}/p_{n+1})\right)$ Since G is DS-diagram, $(\partial B^3)/f_G$ is a special spine for the manifold B^3/f_G, see [4]. Thus, the dual complex of $(\partial B^3)/f_G$ is the singular triangulation of the manifold $M\left(G_{n+g}(g_*, b, q_1/p_1, q_2/p_2, \ldots, q_{n+1}/p_{n+1})\right)$.

Proposition 9.3 *The manifold $M\left(G_{n+g}(g_*, b, q_1/p_1, q_2/p_2, \ldots, q_{n+1}/p_{n+1})\right)$ is homeomorphic to Seifert manifold $S(F_g, b; (p_1, r_1), \ldots, (p_{n+1}, r_{n+1}))$, where r_i is defined in Theorem 8.5*

Proof. The proof is similar to Lemma 5.3. By definition of gluing map φ_i, the boundary of $T(q_i/p_i)$ (or $T(b), M_{*i}$) and torus T_i which are the boundary component of $M_{G_{n+g}}$ are identified such that $\varphi_i(\alpha_i) = \alpha_i$, $\varphi_i(\beta_i) = \beta_i$, $\varphi_i(\gamma_i) = \gamma_i$. Thus, the manifold $M\left(G_{n+g}(g_*, b, q_1/p_1, q_2/p_2, \ldots, q_{n+1}/p_{n+1})\right)$ is homeomorphic to the manifold $M_{G_{n+g}} \cup$

$T(q_i/p_i) \cup T(b) \cup M_{*i}$. By the Theorem 8.5, the manifold $M_{G_{n+g}} \cup T(q_i/p_i) \cup T(b) \cup M_{*i}$ is homeomorphic to Seifert manifold $S(F_g, b; (p_1, r_1), \dots, (p_{n+1}, r_{n+1}))$. ∎

At last, we get the following theorem by Theorem 8.5, Remark 9.2 and Proposition 9.3.

Theorem 9.4 *Let $G := G_{n+g}(g_*, b, q_1/p_1, q_2/p_2, \dots, q_{n+1}/p_{n+1})$ be a DS-diagram of Seifert manifold of $M := B^3/f_G \cong S(F_g, b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_{n+1}, \beta_{n+1}))$. Then, a Singular triangulation of Seifert manifold M is dual complex of $(\partial B^3)/f_G$.*

Example

1. Quaternionic space $\cong M(G_1(1/2, 1/2, 1/2))$
2. Brieskorn manifold
   $\Sigma(2,3,5) \cong S(S^2, -1; (2,1), (3,1), (5,1)) \cong M(G_2(-1, 1/2, 1/3, 1/5))$
3. $S(F_2, -2; (5,3), (3,1), (2,1)) \cong M(G_4(2_*, -2, 2/5, 1/3, 1/2))$

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