箱根セミナ記録 9 3 (1993)55-71.

3次元空間における特異曲面の「切った－貼った」技術

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本研究は、神戸大学理学部・細川藤次教授との共同研究の成果の一部であり、その内容は論文［ＨＳ］において公表準備中である。

３次元多様体、あるいは３次元多様体内の曲面を研究するに際しては、いわゆる「切った-貼った(cut-and-paste)」を日常的に用いるし、またそれなりに有効である。ここでは、この「切った-貼った」技術を特異曲面（つまり曲面の連続写像による像）に対しても適用できるように一般化し、特異曲面をある自然な条件下で変形することを試みる。

またその応用として、３次元空間R^3内の平凡型の絡み目に張った２組の特異円盤系は、４次元上半空間において互いに絡み目ホモトープであることを証明する。

以下の考察は、すべて区分線形のカテゴリーで行う。

§1. Singular loops in a 2-cell

We denote by ∂X and $°X$, respectively, the boundary and the interior of a manifold X. For a subcomplex P in a complex M, by $N(P;M)$ we denote a regular neighborhood of P in M, that is, we construct the second derived of M and take the closed star of P, see [H],[RS].

We shall say that a submanifold X of a manifold Y is **proper** iff $X \cap \partial Y = \partial X$.

By R^n, D^n and S^{n-1} we shall denote the Euclidean n-space, the standard n-cell and the standard (n-1)-sphere ∂D^n, respectively.

1.1.Definition. (1) Let $f : D^1 \to M$ and $g : S^1 \to M$ be non-degenerate continuous maps into a manifold M. Then, the images $f(D^1) = A$ and $g(S^1) = J$ will be called a **singular-arc** (or simply an **arc**) and a **singular-loop** (or simply a **loop**), respectively. In particular, A and J will be called a **simple arc** and a **simple loop**, respectively, if f and g are embeddings. The **boundary** of an arc $f(D^1) = A$ is the image $f(\partial D^1)$ of the boundary ∂D^1, and we denote it by $\partial^* A$.

(2) An arc A in a manifold M is said to be **proper** iff $A \cap \partial F = \partial^* A$. A loop J in a manifold M is said to be **proper** iff $J \subset {}^\circ F$.

(3) Let $B = B_1 \cup \cdots \cup B_n$ be a finite union of proper arcs and proper loops in a 2-manifold F^2. A point p in B is said to be a **singular-point** of maltiplicity k iff the number of the preimage of p is k with $k \geqq 2$.

We shall say that B is **normal**, iff

(i) B has only a finite number of singular-points of multiplicity 2, and

(ii) at every singular point of B, B crosses transversally.

1.2. Lemma. Let $J_1 = J_{11} \cup \cdots \cup J_{1m(1)}$ and $J_2 = J_{21} \cup \cdots \cup J_{2m(2)}$ be finite unions of proper loops in a simply connected 2-manifold F^2 such that $J_1 \cap J_2 = \emptyset$. Then, there exists $j \in \{1, \cdots, m(1)\}$ or $k \in \{1, \cdots, m(2)\}$ so that J_{1j} is contractible in $F^2 - J_2$ or J_{2k} is contractible in $F^2 - J_1$.

Proof. We may assume that $J_1 \cup J_2$ is polygonal and normal.

Let $R = \{R_1, \cdots, R_r\}$ be the set of regions of $F^2 - {}^\circ N(J_1; F^2)$. It will be noticed that $R_1 \cup \cdots \cup R_r \supset J_2$.

If there exist a loop, say J_{2k}, of J_2, and a simply connected region, say R_h, of R with $J_{2k} \subset R_h$, then J_{2k} is contractible in $R_h \subset F^2 - J_1$, and so the proof is complete.

So, we may assume that there exist some non-simply connected regions, say $Q_1, \cdots, Q_q$, of R, so that $Q_1 \cup \cdots \cup Q_q \supset J_2$. Let $C_1 \cup \cdots \cup C_s = \partial Q_1 \cup \cdots \cup \partial Q_q$ be the disjoint union of simple loops on F^2, and let Δ_h be the 2-cell on F^2 with $\partial \Delta_h = C_h$ (h=1, $\cdots$, s). We choose an innermost 2-cell, say Δ_1, in $\{\Delta_1, \cdots, \Delta_s\}$, i.e. there is no other Δ_h in Δ_1. Since Δ_1 is not belong to R and $C_1 = \partial \Delta_1$ is the one of the boundary curves $\partial Q_1 \cup \cdots \cup \partial Q_q$, it holds that $\Delta_1 \cap J_1 \neq \emptyset$, and since Δ_1 does not contain any $Q_1, \cdots, Q_q$ and $J_2 \subset Q_1 \cup \cdots \cup Q_q$, it holds that $\Delta_1 \cap J_2 = \emptyset$. Now, any J_{1j} of J_1 with $J_{1j} \cap \Delta_1 \neq \emptyset$ is contractible in $\Delta_1 \subset F^2 - J_2$, and so the proof is complete. □

By the same way as that of Lemma 1.2, we have the following :

1.3. Theorem. Let $J_i = J_{i1} \cup \cdots \cup J_{im(i)}$ be a finite union of proper loops in a simply connected 2-manifold F^2 for i=1, $\cdots$, μ, such that $J_i \cap J_h = \emptyset$ for $i \neq h$. Then, there exist $j \in \{1, \cdots, \mu\}$ and $k \in \{1, \cdots, m(j)\}$ so that J_{jk} is contractible in $F^2 - \bigcup_{i \neq j} J_i$.

Proof. We prove this by induction on the number μ of the classes J_i. The case of $\mu = 1$ is trivial, and the case $\mu = 2$ is Lemma 1.2. So, we assume that $\mu \geqq 3$ and Theorem is true for $\mu - 1$. We may assume that every J_i is polygonal and normal.

Let $R = \{R_1, \cdots, R_r\}$ be the set of regions of $F^2 - {}^\circ N(J_1;F^2)$. It will be noted that $R_1 \cup \cdots \cup R_r \supset J_2 \cup \cdots \cup J_\mu$.

If there exist a loop, say J_{jk}, of J_j and a simply connected region, say R_h, of R with $J_{jk} \subset R_h$, then $J_i' = J_i \cap R_h$ $(i=2, \cdots, \mu)$ is a finite union of loops in the simply connected region R_h satisfying the conditions of Theorem. By induction hypothesis, we have a loop, say J_{jk}, of $J_j' \subset J_j$ so that J_{jk} is contractible in $R_h - \bigcup_{i \neq 1, j} J_i' \subset F^2 - \bigcup_{i \neq j} J_i$, and so the proof is complete.

So, we may assume that there exist some non-simply connected regions, say $Q_1, \cdots, Q_q$ of R, so that $Q_1 \cup \cdots \cup Q_q \supset J_2 \cup \cdots \cup J_\mu$. Now, the proof of this case, which is omitted here, is the same as that of Lemma 1.2. □

In general, we have the following :

1.4. Theorem. Let $A_i = A_{i1} \cup \cdots \cup A_{in(i)}$ be a finite union of proper arcs in a simply connected 2-manifold F^2 for $i=1, \cdots, \mu$, and let $J_i = J_{i1} \cup \cdots \cup J_{im(i)}$ be a finite union of proper loops in F^2, such that $(A_i \cup J_i) \cap (A_h \cup J_h) = \emptyset$ for $i \neq h$. Then, there exist $j \in \{1, \cdots, \mu\}$ and $k \in \{1, \cdots, m(j)\}$ so that J_{jk} is contractible in $F^2 - \bigcup_{i \neq j} (A_i \cup J_i)$.

Proof. We may assume that every $A_i \cup J_i$ is polygonal and normal. Since every region of $F^2 - {}^\circ N(A_i;F^2)$ is simply connected, the proof of Theorem is similar to that of Theorem 1.3, and so it is omitted here. □

§2. Singular spheres in a 3-cell

In this section, we will discuss singular 2-spheres in a 3-cell and prove similar theorems to these in the previous section.

First let us explain several well-known facts to be used in the sequel.

If a compact 3-manifold M is embeddable in the 3-sphere S^3, then there is a 1-complex G in S^3 such that the exterior $S^3 - {}^\circ N(G;S^3)$ is homeomorphic to M by Fox[F].

A 1-complex G in S^3 is said to be splittable, iff there exists a 2-sphere $S \subset S^3 - G$, such that both components of $S^3 - S$ contain points of G. If a 1-complex $G \subset S^3$ is not splittable, then the exterior $S^3 - {}^\circ N(G;S^3)$ is aspherical, i.e. the second homotopy group $\pi_2(S^3 - {}^\circ N(G;S^3)) = \{0\}$, by Papakyriakopoulos[P].

In particular, if $G \subset S^3$ is a connected 1-complex, then $S^3 - {}^\circ N(G;S^3)$ is aspherical.

We will call a compact 3-manifold M an **aspherical region** iff M is embeddable in S^3 and aspherical.

It holds the following :

2.1. Proposition. (1) If a compact 3-manifold M is embeddable in S^3 and ∂M is connected, then M is an aspherical region.

(2) Let M be an aspherical region with connected boundary ∂M and let $F \subset {}^\circ M$ be a closed connected 2-manifold. Then, there exists an aspherical region R in M with $\partial R = F$. □

The following corresponds to Definition 1.1.

2.2. Definition. (1) Let $f : F^2 \to M$ be a non-degenerate continuous map of compact 2-manifold F^2 into a manifold M. Then, the image $f(F^2) = F$ will be called a **singular-surface**. In particular, singular-surfaces $f(D^2) = D$ and $g(S^2) = S$ will be called a **singular-disk** and a **singular-sphere**, respectively.

The boundary of a singular-surface $f(F^2) = F$ is the image $f(\partial F^2)$, and we denote it by $\partial^* F$.

(2) A singular-surface F in a manifold M is said to be **proper** iff $F \cap \partial M = \partial^* F$.

(3) Let F be a proper singular-surface in a 3-manifold M. A point p in F is a **singular-point** of multiplicity k iff the number of the preimage of p is k with $k \geqq 2$.

We shall say that F is **normal** iff

(i) F has only singular-points of multiplicity 2 and 3,

(ii) the set of singular-points of multiplicity 2 is a finite number of polygonal curves, that is, singular-arcs and singular-loops, which will be called **double-lines**,

(iii) the set of singular-points of multiplicity 3 consists of a finite number of points which are intersection points of the double-lines, which will be called **triple-points**, and

(iv) at every singular-point of multiplicity 2, F crosses transversally.

In fact, every singular-point p of F has one of the neighborhood described in Figure 1, and it is well known that every singular-surface may be ε-approximated by such a normal one.

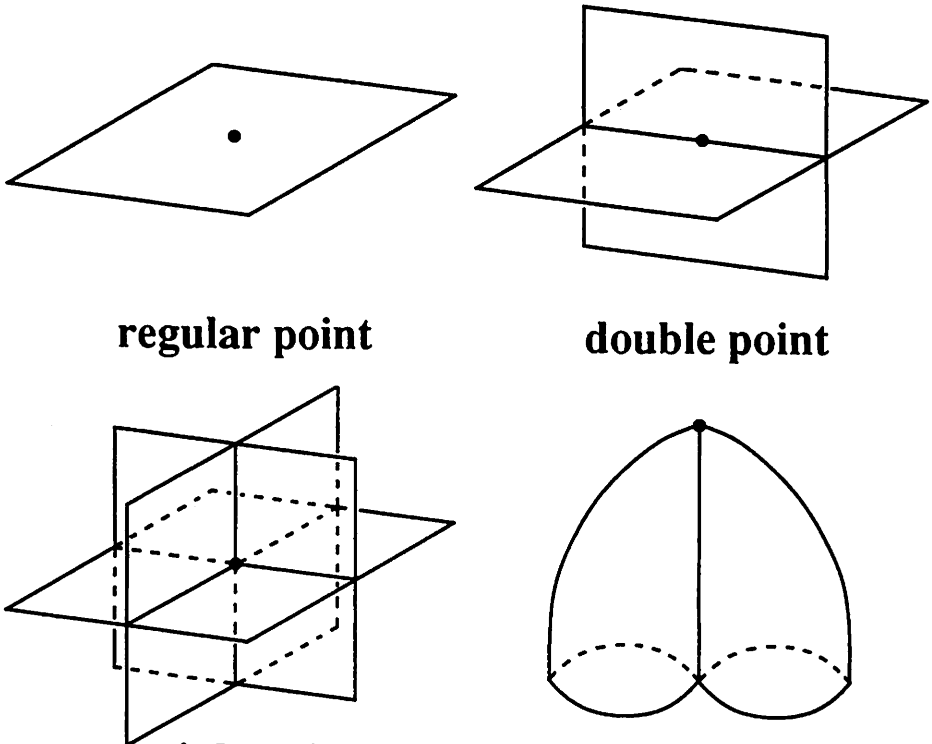

Figure 1

2.3. Lemma. Let $S_1 = S_{11} \cup \cdots \cup S_{1m(1)}$ and $S_2 = S_{21} \cup \cdots \cup S_{2m(2)}$ be finite unions of proper singular-spheres in an aspherical region M with connected boundary ∂M such that $S_1 \cap S_2 = \emptyset$. Then, there exists $j \in \{1, \cdots, m(1)\}$ or $k \in \{1, \cdots, m(2)\}$ so that S_{1j} is contractible in $M - S_2$ or S_{2k} is contractible in $M - S_1$.

Proof. We may assume that $S_1 \cup S_2$ is normal. The proof of this Lemma is similar to that of Lemma 1.2.

Let $\mathbf{R} = \{R_1, \cdots, R_r\}$ be the set of regions of $M - {}^{\circ}N(S_1;M)$. It will be noted that $R_1 \cup \cdots \cup R_r \supset S_2$.

If there exist a singular-sphere, say S_{2k}, of S_2 and an aspherical region, say R_h, of $\mathbf{R}$ with $S_{2k} \subset R_h$, then S_{2k} is contractible in $R_h \subset M - S_1$, and completing the proof.

So, we may assume that there exist some spherical regions, say $Q_1, \cdots, Q_q$, in $\mathbf{R}$, so that $Q_1 \cup \cdots \cup Q_q \supset S_2$. Let $F_1 \cup \cdots \cup F_s = \partial Q_1 \cup \cdots \cup \partial Q_q$ be the disjoint union of closed connected 2-manifolds, and let M_h be the aspherical region in M with $\partial M_h = F_h$ for $h=1, \cdots, s$, see Proposition 2.1(2). We choose an innermost region, say M_1, in these aspherical regions, that is, there are no other M_h in M_1. Then, by the same way as the proof of Lemma 1.2, it is easily checked that $M_1 \cap S_1 \neq \emptyset$ and $M_1 \cap S_2 = \emptyset$. Now, any S_{1j} of S_1 with $S_{1j} \cap M_1 \neq \emptyset$ is contractible in $M_1 \subset M - S_2$, and completing the proof. □

The following theorems correspond to Theorems 1.3 and 1.4, respectively.

2.4. Theorem. Let $S_i = S_{i1} \cup \cdots \cup S_{im(i)}$ be a finite union of proper singular-spheres in an aspherical region M with connected boundary ∂M for $i = 1, \cdots, \mu$, such that $S_i \cap S_h = \emptyset$ for $i \neq h$. Then, there exist $j \in \{1, \cdots, \mu\}$ and $k \in \{1, \cdots, m(j)\}$ so that S_{jk} is contractible in $M - \bigcup_{i \neq j} S_i$.

Proof. The proof is similar to that of Lemma 2.3, and is word for word that of Theorem 1.3. □

2.5. Theorem. Let M be an aspherical region with connected boundary ∂M. Let $D_i = D_{i1} \cup \cdots \cup D_{in(i)}$ and $S_i = S_{i1} \cup \cdots \cup S_{im(i)}$ be finite unions of proper singular-disks and proper singular-spheres in M, respectively, for $i=1,\cdots,\mu$, such that $(D_i \cup S_i) \cap (D_h \cup S_h) = \emptyset$ for $i \neq h$. Then, there exist $j \in \{1,\cdots,\mu\}$ and $k \in \{1,\cdots,m(j)\}$, so that S_{jk} is contractible in $M - \bigcup_{i \neq j} (D_i \cup S_i)$.

Proof. We may assume that every $D_i \cup S_i$ is normal. Since every region R of $M - {}^\circ N(D_i;M)$ is an aspherical region, the proof of this Theorem is similar to that of Theorem 2.4, and is word for word that of Theorem 1.4. □

§3. Singular cut-and-pastes

3.1. Definition. Let M^3 be a 3-manifold, and let E^2 be a 2-manifold in ${}^\circ M^3$. Let $f : F^2 \to M^3$ be a non-degenerate continuous map of a compact 2-manifold F^2 into M^3 such that

(i) $f(F^2) = F$ is a normal singular-surface,

(ii) F intersects with E^2 transversally, and

(iii) every triple-point and every branch point of F do not lie on E^2.

Then, the intersection $F \cap E^2$ consists of a finite number of arcs and loops. Let J be a loop in $F \cap E^2$, and let J^* be the preimage of J in F^2; J^* is a simple loop. We suppose that J^* is 2-sided on F^2, and let F'^2 be the 2-manifold obtained from F^2 by attaching a 2-handle along J^*; $F'^* = F^2 \cup h^2$.

Now, we suppose that J is contractible on E^2. Then, we have a non-degenerate continuous map, say g, of D^2 into $E^2 \subset M^3$ such that $g(\partial D^2) = J$. Using the product structure $N(E^2;M^3) =$

$E^2 \times D^1$, we define a non-degenerate continuous map $f' : F'^2 \to M^3$ as follows :

$$f' \mid F'^2 - h^2(D^2 \times \partial D^1) = f \mid F^2 - h^2(\partial D^2 \times D^1),$$
$$f' \mid h^2(D^2 \times \partial D^1) = g \times \partial D^1.$$

We say that $F' = f'(F'^2)$ is obtained from $F = f(F^2)$ by a **cut-and-paste** along $J \subset E^2$, and we denote simply by $F \to F'$.

It will be noticed that $F' \cap E^2 = F \cap E^2 - J$ and that $F'^2 = D^2 \amalg S^2$ (a disjoint union) provided that $F^2 = D^2$ and $F'^2 = S^2 \amalg S^2$ provided that $F^2 = S^2$.

3.2. Theorem. Let $O_i = O_{i1} \cup \cdots \cup O_{in(i)}$ be a trivial link in the 3-sphere S^3 (or the 3-space R^3) for $i=1,\cdots,\mu$, such that $O_1 \cup \cdots \cup O_\mu$ is also a trivial link. Let $D_i = D_{i1} \cup \cdots \cup D_{in(i)}$ be a finite union of normal singular-disks in S^3 for $i=1,\cdots,\mu$, such that $\partial^* D_{ij} = O_{ij}$ for $i=1,\cdots,\mu$ and $j=1,\cdots,n(i)$, and $D_i \cap D_h = \emptyset$ for $i \neq h$.

Let $D^*_i = D^*_{i1} \cup \cdots \cup D^*_{in(i)}$ be mutually disjoint 2-cells in S^3 (or R^3) for $i=1,\cdots,\mu$, such that $\partial D^*_{ij} = O_{ij}$ for $i=1,\cdots,\mu$ and $j=1,\cdots,n(i)$, and $D^*_i \cap D^*_h = \emptyset$ for $i \neq h$.

We suppose that $D_1 \cup \cdots \cup D_\mu$ intersects with $D^*_1 \cup \cdots \cup D^*_\mu$ transversally, and any triple-point and any branch-point of $D_1 \cup \cdots \cup D_\mu$ do not lie on $D^*_1 \cup \cdots \cup D^*_\mu$.

Then, there exists a finite sequence of cut-and-pastes

$$D_1 \cup \cdots \cup D_\mu = D_1^{(0)} \cup \cdots \cup D_\mu^{(0)} \to D_1^{(1)} \cup \cdots \cup D_\mu^{(1)} \to \cdots$$
$$\to D_1^{(u)} \cup \cdots \cup D_\mu^{(u)} \to \cdots \to D_1^{(w)} \cup \cdots \cup D_\mu^{(w)}$$

along $(D_1 \cup \cdots \cup D_\mu) \cap (D^*_1 \cup \cdots \cup D^*_\mu) \subset D^*_1 \cup \cdots \cup D^*_\mu$ such that

(1) $D_i^{(u)} = D_{i1}^{(u)} \cup \cdots \cup D_{in(i)}^{(u)} \cup S_{i1}^{(u)} \cup \cdots \cup S_{im(i)}^{(u)}$, where

$D_{ij}{}^{(u)}$ is a singular-disk with $\partial^* D_{ij}{}^{(u)} = O_{ij}$ and $S_{is}{}^{(u)}$ is a singular-sphere, for $i=1,\cdots,\mu$; $j=1,\cdots,n(i)$; $u=1,\cdots,w$; $s=1,\cdots,m(i)$,

(2) $D_i{}^{(u)} \cap D_h{}^{(u)} = \varnothing$ for $i \neq h$, $u=1,\cdots,w$, and

(3) $D_i{}^{(w)} \cap D^*_h = \varnothing$ for $i \neq h$, and $D_i{}^{(w)} \cap D^*_i = (D_{i1}{}^{(w)} \cup \cdots \cup D_{in(i)}{}^{(w)}) \cap D^*_i$ consists of a finite number of proper arcs in D^*_i.

Proof. From our hypothesis, $D_{ij} \cap D^*_{hk}$ consists of proper loops in D^*_{hk} provided that $i \neq h$, and $D_{ij} \cap D^*_{ik}$ consists of proper loops and proper arcs in D^*_{ik} for every i, j, k. Therefore, by the induction on the number $n = n(1)+\cdots+n(\mu)$ of 2-cells in $D^*_1 \cup \cdots \cup D^*_\mu$, it suffices to show that there exists a finite sequence of cut-and-pastes of $D_1 \cup \cdots \cup D_\mu$ along proper loops $(D_1 \cup \cdots \cup D_\mu) \cap D^*_{11} \subset D^*_{11}$ so that $D_1{}^{(u)} \cup \cdots \cup D_\mu{}^{(u)}$ satisfies the conditions (1), (2) and

(3) $D_i{}^{(w)} \cap D^*_{11} = \varnothing$ and $D_i{}^{(w)} \cap D^*_{1j} = D_i{}^{(w)} \cap D^*_{1j}$ for $i=2,\cdots,t$ and $j=2,\cdots,n(1)$, and $D_1{}^{(w)} \cap D^*_{11}$ consists of a finite number of proper arcs in D^*_{11} and $D_1{}^{(w)} \cap D^*_{1j} = D_1{}^{(w)} \cap D^*_{1j}$ for $j=2,\cdots,n(1)$.

We consider $D_1 \cup \cdots \cup D_\mu$ and D^*_{11}. Let $A_1 = A_{11} \cup \cdots \cup A_{1a(1)}$ be the collection of proper arcs in $D_1 \cap D^*_{11}$ on D^*_{11}, and let $A_i = \varnothing$ be the collection of proper arcs in $D_i \cap D^*_{11}$ for $i=2,\cdots,\mu$. Let $J_i = J_{i1} \cup \cdots \cup J_{ib(i)}$ be a collection of proper loops in $D_i \cap D^*_{11}$ on D^*_{11} for $i=1,\cdots,\mu$. Then, $A_i \cup J_i$ satisfies the assumptions in Theorem 1.4, and so there exists a loop J_{jk} of some J_j such that J_{jk} is contractible in $D^*_{11} - \bigcup_{i \neq j} (A_i \cup J_i)$. We have a non-degenerate continuous map $g : D^2 \to D^*_{11}$ such that $g(D^2) \cap (A_i \cup J_i) = \varnothing$ for $i \neq j$. Using this g, we perform the first cut-and-paste for $D_j \subset D_1 \cup \cdots \cup D\mu = D_1{}^{(0)} \cup \cdots \cup D\mu^{(0)}$ and obtain $D_1{}^{(1)} \cup \cdots \cup D\mu^{(1)}$. Let w be the number of loops in $(D_1 \cup \cdots \cup D\mu) \cap D^*{}_{11}$. By the repetition of the procedure w times, we can get rid of all loops in $(D_1 \cup \cdots \cup D\mu) \cap D^*{}_{11}$, and it is easily checked that $D_1{}^{(u)} \cup \cdots \cup D\mu^{(u)}$ satisfies the required conditions for u=1, ⋯, w, and we complete the proof of Theorem. □

3.3. **Remarks.** (1) From the proof of Theorem 3.2, we know that w is the number of loops in $(D_1 \cup \cup D\mu) \cap (D^*{}_1 \cup \cup D^*\mu)$ and $w = m(1) + \cdots + m(\mu)$, which is the number of singular-spheres in $D_1{}^{(w)} \cup \cdots \cup D\mu^{(w)}$.

(2) Let D and D^* be a normal singular-disk and a 2-cell, respectively, in S^3 (or R^3) such that $\partial^* D = \partial D^* = 0$ (a trivial knot). Let A be a proper arc of $D \cap D^*$ in D^* and let α be a simple arc in 0 with $\partial \alpha = \partial^* A$. Since $A \cup \alpha$ is contractible in D^*, we can formulate a cut-and-paste of D along $A \cup \alpha \subset D^*$ as the same way as Definition 3.1 except for obvious modefications, so that $D \rightarrow D' = D_1' \cup S_1'$, where S_1' is a singular-sphere and D_1' is a singular-disk with $\partial^* D_1' = 0$.

Now, in the notation and assumptions of Theorem 3.2, we suppose that $D_{ij} \cap D^*{}_{ik}$ does not contain proper arcs on $D^*{}_{ik}$ for i=1, ⋯, μ and $j \neq k$. Then, we can remove proper arcs of $D_i{}^{(w)} \cap D^*{}_i$ by a finite sequence of the modified cut-and-pastes.

§4. Applications to link theory

A continuous image of the 3-cell D^3 will be called a **singular-ball**. The **boundary** of a singular-ball B is the image of ∂D^3, and we denote it by $\partial^* B$.

We use here the same notation as that of Section 0 in [KSS]. The following is an generalization of Horibe-Yanagawa's Lemma [KSS, Lemma 1.6] in a sense.

4.1. Theorem. In the notation and assumptions of Theorem 3.2, let $\Sigma_i = \Sigma_{i1} \cup \cdots \cup \Sigma_{in(i)}$ be a finite union of singular-spheres in $R^3[0,1]$ defined by

$$\Sigma_{ij} = D_{ij}[0] \cup O_{ij} \times [0,1] \cup D^*_{ij}[1]$$

for $i=1,\cdots,\mu$ and $j=1,\cdots,n(i)$. Then, we can find a finite union of singular-balls $\mathbf{B}_i = B_{i1} \cup \cdots \cup B_{in(i)}$ in $R^3[0,\infty)$ for $i=1,\cdots,\mu$, such that $\partial^* B_{ij} = \Sigma_{ij}$ for every i and j, and $\mathbf{B}_i \cap \mathbf{B}_h = \emptyset$ for $i \neq h$.

Proof. The proof is similar to that of [KSS, Lemma 1.6]. We shall construct the required singular-balls $\mathbf{B}_1 \cup \cdots \cup \mathbf{B}_\mu$ by specifying the cross-sections $B_{ij} \cap R^3[t]$ for all i and j.

Under the notation of Theorem 3.2, We also use Theorem 3.2. Let g_u: $D^2 \to D^*_1 \cup \cdots \cup D^*_\mu$ $(u=1,\cdots,w)$ be a non-degenerate continuous map such that we perform the u-th cut-and-paste

$$\mathbf{D}_1^{(u-1)} \cup \cdots \cup \mathbf{D}_\mu^{(u-1)} \Rightarrow \mathbf{D}_1^{(u)} \cup \cdots \cup \mathbf{D}_\mu^{(u)}$$

in Theorem 3.2 along the loop $g_u(\partial D^2)$ under g_u. We extend g_u to a continuous map

$$g^\#_u: h^2(D^2 \times D^1) \to N(\mathbf{D}^*_1 \cup \cdots \cup \mathbf{D}^*_\mu ; R^3) = (\mathbf{D}^*_1 \cup \cdots \cup \mathbf{D}^*_\mu) \times D^1$$

of the 3-cell $h^2(D^2 \times D^1)$ naturally, and we denote the singular-ball $g^\#_u(h^2(D^2 \times D^1))$ by H_u. We divide the interval $[0,1]$ into the subintervals $[0,t_1],[t_1,t_2],\cdots,[t_{w-1},t_w],[t_w,1]$, where $t_u = u/(w+1)$, $u=1,\cdots,w$. Let

$$(\mathbf{B}_1 \cup \cdots \cup \mathbf{B}_\mu) \cap R^3[t] = (\mathbf{D}_1 \cup \cdots \cup \mathbf{D}_\mu)[t] \text{ for } 0 \leqq t < t_1$$

$$(\mathbf{B}_1 \cup \cdots \cup \mathbf{B}_\mu) \cap R^3[t_1] = (\mathbf{D}_1 \cup \cdots \cup \mathbf{D}_\mu \cup H_1)[t_1],$$

$$(B_1 \cup \cdots \cup B\mu) \cap R^3[t] = (D_1{}^{(1)} \cup \cdots \cup D\mu{}^{(1)})[t] \quad \text{for } t_1 < t < t_2,$$

$$\cdots\cdots\cdots$$

$$(B_1 \cup \cdots \cup B\mu) \cap R^3[t] = (D_1{}^{(u-1)} \cup \cdots \cup D\mu{}^{(u-1)})[t] \quad \text{for } t_{u-1} < t < t_u,$$

$$(B_1 \cup \cdots \cup B\mu) \cap R^3[t_u] = (D_1{}^{(u-1)} \cup \cdots \cup D\mu{}^{(u-1)} \cup H_u)[t_u],$$

$$(B_1 \cup \cdots \cup B\mu) \cap R^3[t] = (D_1{}^{(u)} \cup \cdots \cup D\mu{}^{(u)})[t] \quad \text{for } t_u < t < t_{u+1},$$

$$\cdots\cdots\cdots$$

$$(B_1 \cup \cdots \cup B\mu) \cap R^3[t_w] = (D_1{}^{(w-1)} \cup \cdots \cup D\mu{}^{(w-1)} \cup H_w)[t_w],$$

$$(B_1 \cup \cdots \cup B\mu) \cap R^3[t] = (D_1{}^{(w)} \cup \cdots \cup D\mu{}^{(w)})[t] \quad \text{for } t_w < t \leqq 1.$$

Thus, we constructed $(B_1 \cup \cdots \cup B\mu) \cap R^3[0,1]$ which consists of $n=n(1)+\cdots+n(\mu)$ singular-balls with $w=m(1)+\cdots+m(\mu)$ singular-balls removed.

Let $S_{ij}{}^{(w)} = D_{ij}{}^{(w)} \cup D^*{}_{ij}$ be the singular-sphere for $i=1,\cdots,\mu$ and $j=m(i)+1,\cdots,m(i)+n(i)$, and let $S_i = D_i{}^{(w)} \cup D^*{}_i = S_{i1}{}^{(w)} \cup \cdots \cup S_{im(i)+n(i)}{}^{(w)}$, which consists of $m(i)+n(i)$ singular-spheres in R^3. From Theorem 3.2(2) and (3), it is easy to see that $S_i \cap S_h = \varnothing$ for $i \neq h$, which is the assumption of Theorem 2.4.

We divide the interval $[1,2]$ into the $n+w+1$ subintervals $[1,s_1]$, $[s_1,s_2],\cdots,[s_{n+w-1},s_{n+w}],[s_{n+w},2]$, where $s_v = 1 + v/(n+w+1)$, $v=1,\cdots,n+w$. From now on, we construct $(B_1 \cup \cdots \cup B\mu) \cap R^3[1,2]$ so that $(B_1 \cup \cdots \cup B\mu) \cap R^3[0,2]$ forms the required singular-balls. By Theorem 2.4, there exist $j \in \{1,\cdots,\mu\}$ and $k \in \{1,\cdots,m(j)+n(j)\}$ so that $S_{jk}{}^{(w)}$ is contractible in $R^3 - \bigcup_{i \neq j} S_i$. Let $g_1: D^3 \to R^3 - \bigcup_{i \neq j} S_i$ be a continuous map such that $g_1(\partial D^3) = S_{jk}{}^{(w)}$, and we denote $g_1(D^3)$ by E_1. We set $S_j{}^{(1)} = S_j - S_{jk}{}^{(w)}$, and $S_i{}^{(1)} = S_i$ for $i \neq j$. Then, we define $(B_1 \cup \cdots \cup B\mu) \cap R^3[1,s_2)$ as follows :

$$(B_1 \cup \cdots \cup B\mu) \cap R^3[t] = (S_1 \cup \cdots \cup S\mu)[t] \quad \text{for } 1 \leqq t < s_1,$$

$(B_1 \cup \cdots \cup B\mu) \cap R^3[s_1] = (S_1 \cup \cdots \cup S\mu \cup E_1)[s_1]$,

$(B_1 \cup \cdots \cup B\mu) \cap R^3[t] = (S_1{}^{(1)} \cup \cdots \cup S\mu^{(1)})[t]$ for $s_1 < t < s_2$.

By Theorem 2.4, there exist $j' \in \{1, \cdots, \mu\}$ and $k' \in \{1, \cdots, m(j') + n(j')\}$ so that $S_{j'k'}$ is contractible in $R^3 - \bigcup_{i \neq j} S_i{}^{(1)}$. Let $g_2 \colon D^3 \to R^3 - \bigcup_{i \neq j} S_i{}^{(1)}$ be a continuous map with $g_2(\partial D^3) = S_{j'k'}$, and we denote $g_2(D^3)$ by E_2. We set $S_{j'}{}^{(2)} = S_{j'}{}^{(1)} - S_{j'k'}{}^{(1)}$ and $S_i{}^{(2)} = S_i{}^{(1)}$ for $i \neq j'$. We define $(B_1 \cup \cdots \cup B\mu) \cap R^3[s_2, s_3)$ as follows :

$(B_1 \cup \cdots \cup B\mu) \cap R^3[s_2] = (S_1{}^{(1)} \cup \cdots \cup S\mu^{(1)} E_2)[s_2]$,

$(B_1 \cup \cdots \cup B\mu) \cap R^3[t] = (S_1{}^{(2)} \cup \cdots \cup S\mu^{(2)})[t]$ for $s_2 < t < s_3$.

For $R^3[s_3, s_4), \cdots, R^3[s_{n+w-1}, s_{n+w}), R^3[s_{n+w}, 2)$, we repeat this process. It should be noticed that $S_1{}^{(n+w-1)} \cup \cdots \cup S\mu^{(n+w-1)}$ consists of a single singular-sphere and $S_1{}^{(n+w)} \cup \cdots \cup S\mu^{(n+w)} = \emptyset$. Therefore, $(B_1 \cup \cdots \cup B\mu) \cap R^3[s_{n+w}]$ consists of a singular-ball $E_{n+w}[s_{n+w}]$, and $(B_1 \cup \cdots \cup B\mu) \cap R^3[t] = \emptyset$ for $s_{n+w} < t < 2$.

Thus, we obtain a union of singular-balls $B_i = B_{i1} \cup \cdots \cup B_{in(i)}$ in $R^3[0, \infty)$ for $i = 1, \cdots, \mu$ such that $\partial^* B_{ij} = \Sigma_{ij}$. From our construction, it is easily checked that $B_i \cap B_h = \emptyset$ for $i \neq h$, and this completes the proof of Theorem 4.1. □

The relation of link-homotopy was introduced in classical link theory by Milnor[M], and studied in higher dimensional links by Massey-Rolfsen[MR] and Koschorke[K], etc. We record a corollary to Theorem 4.1 involving in link-homotopy.

4.2. Definition. Let $P_1, \cdots, P\mu$ be polyhedra, and let $P = P_1 \amalg \cdots \amalg P\mu$ be the disjoint union, and let X be a manifold. A continuous map $f : P \to X$ is said to be a **link-map**, iff $f(P_i) \cap f(P_h) = \emptyset$ for $i \neq h$. Two link-maps f_0 and f_1 of P into X will be called **link-homotopic**, iff there exists a homotopy $\{\eta_t\}_{t \in I} : P \to X$ such that $\eta_0 = f_0$, $\eta_1 = f_1$, and $\eta_t(P_i) \cap \eta_t(P_h) = \emptyset$ for $i \neq h$ and each $t \in I = [0,1]$.

4.3. Theorem. Let $O_i = O_{i1} \cup \cdots \cup O_{in(i)}$ be a trivial link in the 3-space $R^3 = R^3[0] \subset R^3[0,\infty)$ (or $S^3 \subset \partial D^4$) for $i=1,\cdots,\mu$, such that $O_1 \cup \cdots \cup O_\mu$ is also a trivial link. Let $P_i = D^2_{i1} \amalg \cdots \amalg D^2_{in(i)}$ be the disjoint union of $n(i)$ 2-cells for $i=1,\cdots,\mu$, and we set $P = P_1 \amalg \cdots \amalg P_\mu$. Let f and e be non-degenerate link-maps of P into R^3 (or S^3) such that $f(\partial D^2_{ij}) = O_{ij} = e(\partial D^2_{ij})$ for $i=1,\cdots,\mu$ and $j=1,\cdots,n(i)$.

Then, f and e are link-homotopic in $R^3[0,\infty)$ (or D^4) keeping $O_1 \cup \cdots \cup O_\mu$ fixed.

Proof. Let $f(D^2_{ij}) = D_{ij}$ and $D_i = D_{i1} \cup \cdots \cup D_{in(i)}$ for $i=1,\cdots,\mu$ and $j=1,\cdots,n(i)$. Let $g : P \to R^3$ be an embedding, and let $g(D^2_{ij}) = D^*_{ij}$ and $D^*_i = D^*_{i1} \cup \cdots \cup D^*_{in(i)}$. In this notation, it suffices to show that f and g are link-homotopic in $R^3[0,\infty)$ keeping $O_1 \cup \cdots \cup O_\mu$.

In the notation of Theorem 4.1, we have a finite union of singular-balls $B_1 \cup \cdots \cup B_\mu$, $B_i = B_{i1} \cup \cdots \cup B_{in(i)}$ in $R^3[0,\infty)$ such that $B_i \cap B_h = \emptyset$ for $i \neq h$ and $\partial^* B_{ij} = \Sigma_{ij}$. Let $b_{ij}: D^2 \times I \to R^3[0,\infty)$ be a continuous map of the 3-cell $D^2 \times I$ such that $b_{ij}(D^2 \times I) = B_{ij}$. We may assume that $b_{ij} | D^2 \times 0 = f | D^2_{ij}$ and $b_{ij} | D^2 \times 1 = g | D^2_{ij}$. Then, associating with these b_{ij}, we have a link-homotopy $\{\eta_t\}_{t \in I} : P \to R^3[0,\infty)$ defined by

$$\eta_t(D^2_{ij}) = b_{ij}(D^2 \times t)$$

for every $t \in I$. From the condition of the singular-balls $B_1 \cup \cdots \cup B_\mu$ in Theorem 4.1, it is easily checked that this homotopy $\{\eta_t\}_{t \in I}$ between f and g satisfies our required condition, and completing the proof of Theorem 4.3. □

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