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Milnor Move and Self Δ-Equivalence for Links

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Abstract

It is known that there are links such that they are cobordant but not self Δ-equivalent. The Milnor move defined in this paper keeps the cobordism of links. Furthermore there are M-equivalent links (namely they are deformed to each other by a finite sequence of Milnor moves) which are not self Δ-equivalent. However by adding some condition to these link, we will prove the following: If two links are strongly M-equivalent, they are self Δ-equivalent.

1 Introduction.

All links will be assumed to be ordered and oriented and they will be considered up to ambient isotopy in a 3-space R^3.

A *Δ-move*,[2], is a local move of links as illustrated in Fig. 1. If three strands in Fig. 1 belong to a same component of a link, we say it *a self Δ-move*,[4],[5]. For two links ℓ and ℓ', ℓ is said to be *self Δ-equivalent to ℓ'* if ℓ can be deformed into ℓ' by a finite sequence of self Δ-move. Especially if ℓ' is a trivial link, we say that ℓ is *self Δ-equivalent to* 0.

Fig. 1

Another local moves called *a Milnor move* for knots is defined in [1]. In this paper, we deal a Milnor move for links. Namely, let B^3 be a 3-ball such that $\ell \cap B^3$ is the tangle as illustrated in Fig. 2. The deformation from Fig. 2(a) to 2(b) is called *an M_m^+-move* and that from Fig. 2(b) to 2(a) is called *an M_m^--move* and B^3 is called a 3-ball *associated to this move.* Especially, for a $M_m^{\pm}$-move and a 3-ball B^3 associated to this move, if there is a knot k of ℓ such that $\sharp(k \cap B^3) \geq 2$, the move is called *a strong $M_m^{\pm}$-move*, where $\sharp(x)$ means the number of connected components of x. Furthermore *an M_m* (or *a strong M_m-move*) means either an

M_m^+ (resp. a strong M_m^+)-move or an M_m^- (resp. a strong M_m^-)-move. *A Milnor* (or *a strong Milnor*)*-move* means an M_m (resp. a strong M_m)-move for any integer $m(\geq 2)$.

Fig. 2

For two links ℓ and ℓ', ℓ is said to be *M-equivalent to* ℓ' if ℓ can be deformed into ℓ' by a finite sequence of Milnor moves. Especially, if ℓ can be deformed into ℓ' by a finite sequence of strong Milnor moves, we say that ℓ is *strongly M-equivalent to* ℓ'.

In this paper, we consider self Δ-equivalence for strongly M-equivalent links.

Although self Δ-equivalence does not imply the cobordism of links,[3],[4], we easily see that M-equivalence implies cobordism. For two links ℓ, ℓ' as illustrated in Fig. 3, ℓ is M-equivalent to ℓ'. But it is known that ℓ is not self Δ-equivalent to ℓ' by [4]. With respect to this example, $\sharp((\text{each component of } \ell) \cap B^3) \leq 1$ for the 3-ball B^3 associated to an M_2-move. But for strongly M-equivalence, we will prove the following.

Fig. 3

Theorem. *For two links ℓ and ℓ', if ℓ is strongly M-equivalent to ℓ', then ℓ is self Δ-equivalent to ℓ'.*

2 Self Δ-equivalence for strongly M-equivalent links.

Let $\mathcal{L}$ be an n-component link and $\mathcal{B} = B_1 \cup ... \cup B_p$ a disjoint union of bands such that $L \cap B_i = L \cap \partial B_i$ which consists of two arcs orientation coherently for each i. Then we obtain a link $\mathcal{L}' = \mathcal{L} \oplus \partial \mathcal{B}$, where $\oplus$ means the homological addition. If $\sharp(\mathcal{L}') = n + p$ for a positive integer p, $\mathcal{L}'$ is said to be obtained *by a fission of* $\mathcal{L}$ or $\mathcal{L}$ is said to be obtained *by a fusion* of $\mathcal{L}'$. Especially $\mathcal{L}$ is said to be obtained *by a fusion of* $\mathcal{L}'_1$ and $\mathcal{L}'_2$ for $\mathcal{L}' = \mathcal{L}'_1 \cup \mathcal{L}'_2$ if each band B_i of $\mathcal{B}$ connects an arc of $\mathcal{L}'_1$ and one of $\mathcal{L}'_2$.

Now let us prove Theorem.

Proof of Theorem. Since ℓ is stongly M-equivalent to ℓ', there is a sequence of links, say $\ell_0(= \ell), \ell_1, ..., \ell_r(= \ell')$ such that ℓ_{i+1} is obtained by a strong $M_{m_i}^+$ (or a strong $M_{m_i}^-$)-move of ℓ_i for $i = 0, ..., r-1$ and $m_i(\geq 2)$. Therefore, to prove

Theorem, it is sufficient to do that ℓ' is obtained by a strong M_m^+-move of ℓ. Hence ℓ is obtained by a fusion of ℓ' and a trivial link $\mathcal{O} = O_1 \cup ... \cup O_n$ as illustrated in Fig. 4.

Fig. 4

Let $\mathcal{B} = B_1 \cup ... \cup B_m$ be a union of mutually disjoint bands of fusion of ℓ' and $\mathcal{O}$ and $\mathcal{D} = D_1 \cup ... \cup D_m$ mutually disjoint disks with $\partial\mathcal{D} = \mathcal{O}$, Fig. 4. Since ℓ' is obtained by a strong M_m^+-move of ℓ, there is a knot, say k, of ℓ such that $\sharp(k \cap B^3) \geq 2$. Then there are two integers $i, i+j (1 \leq i \leq m-1, j \geq 1)$ such that $k \cap B_i \neq \emptyset$ and $k \cap B_{i+j} \neq \emptyset$. By deforming D_i, D_{i+j} suitably in B^3, let us prove that there are two mutually disjoint non-singular disks E_i, E_{i+j} in B^3 with $\partial E_i = O_i, \partial E_{i+j} = O_{i+j}$ such that $int.E_i \cap (\ell \cup \mathcal{B}) = int.E_i \cap B_{i+j}$ and $int.E_{i+j} \cap (\ell \cup \mathcal{B}) = int.E_{i+j} \cap B_i$ both of which consist of arcs of ribbon type.

If $j = 1$, we put $E_i = D_i$. Next we consider the case that $j > 1$. For any integer $h(1 \leq h \leq j-1)$, take 2^{h-1} disjoint 2-spheres $\Sigma_h^1, ..., \Sigma_h^{2^{h-1}}$ each of which is parallel to $\partial N(D_{i+h} : R^3)$, Fig. 5(a). For $\Sigma_{h-1}^1, ..., \Sigma_{h-1}^{2^{h-2}}$ and $\Sigma_h^1, ..., \Sigma_h^{2^{h-1}}$, we attach 2^{h-1} parallel tubes $T_h^1, ..., T_h^{2^{h-1}}$ along B_{i+h} as illustrated in Fig. 5(b), where T_1^1 connects D_i and Σ_1^1, Fig. 5(b). Now by a cut and paste of D_i, we may easily construct a non-sigular disk E_i satisfying the above conditions.

Fig. 5

By the same construction with that of E_i, we obtain a non-singular disk E_{i+j} satisfying the above conditions and $E_i \cap E_{i+j} = \emptyset$.

Now let $E_i \cap B_{i+j}$ and $E_{i+j} \cap B_i$ consist of arcs of ribbon type, say $\alpha_1, ..., \alpha_r$ and $\beta_1, ..., \beta_s$ respectively. Since $int.E_i \cap (\ell \cup \mathcal{B}) = \alpha_1 \cup ... \cup \alpha_r$ and $\partial\alpha_i \subset k$, the deformation, which is the elmination of β_1 from FIg. 6(a) to 6(b) can be realized by a finite sequence of self Δ-moves applying k,[5], where β_1 is nearest to $\partial B_i \cap \partial E_i$ among $E_i \cap B_{i+j}$ on B_i.

Fig. 6

By doing the above successively, we may eliminate $\beta_2, , , , \beta_s$. As a result, the link obtained by the above deformation is ambient isotopic to ℓ'. Hence we obtain that ℓ is self Δ-equivalent to ℓ'.

For two n-component links ℓ and ℓ', if ℓ can be deformed into ℓ' by an M_m-move for $m > n$. Then the M_m-move is a strong M_m-move. Hence we obtain the following by Theorem.

Corollary. *For two n-component links ℓ and ℓ', suppose that ℓ can be deformed into ℓ' by a finite sequence of Milnor moves, say an M_{m_1}-,..., and an M_{m_r}-move. If $m_i > n$ for each $i = 1, ..., r$, then ℓ is self Δ-equivalent to ℓ'.*

Remark. By a recent result in [6], it is known that, if ℓ is cobordant to 0, it is self Δ-equivalent to 0. Therefore if ℓ is M-equivalent to 0, ℓ is cobordant to 0 and so ℓ is self Δ-equivalent to 0.

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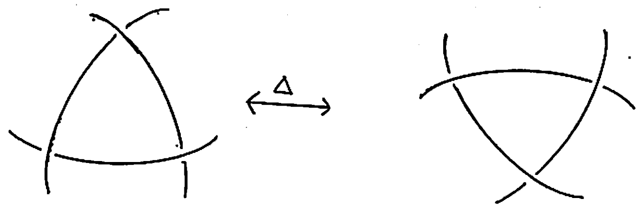

Fig. 1

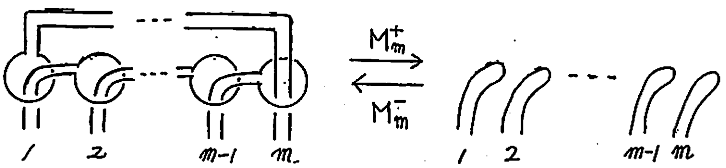

Fig. 2

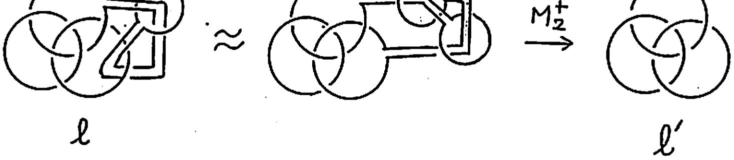

Fig. 3

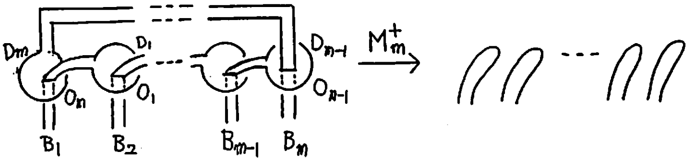

Fig. 4

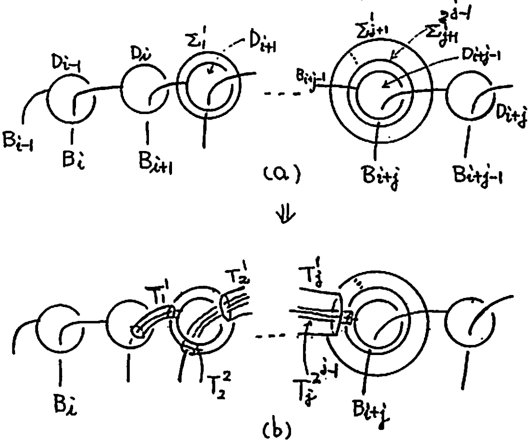

Fig. 5

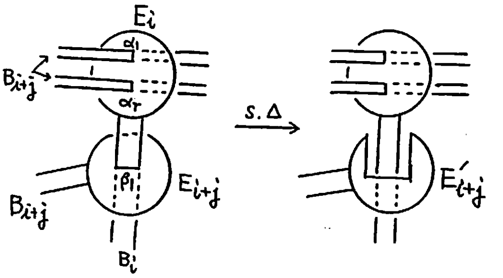

Fig. 6