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On h-split links which are not Δ-split links

Tetsuo SHIBUYA (Osaka Institute of Technology)

1 Introduction.

Throughout this paper, knots and links are oriented in an oriented 3-space R^3. Let $\mathcal{D} = D_1 \cup ... \cup D_n$ be a union of mutually disjoint disks in R^3. For a link $\ell(= \partial\mathcal{D}), \ell$ is called *a Δ-split link* if the set of singularities of $\mathcal{D}$, denoted by $\mathcal{S}(\mathcal{D})$, does not have an arc of clasp type and D_i is called *a claspless disk.* ℓ is called *an h-split link* if $\mathcal{S}(\mathcal{D})$ consists of mutually disjoint arcs of clasp type each of which is *simple*, namely not self intersection,([2]).

Any singularity of $\mathcal{S}(\mathcal{D})$ can be deformed into arcs of simple clasp type by an ambient isotopy of R^3, each Δ-split link is an h-split link.

In this paper, we will prove that the converse does not hold.

Theorem. *There is an h-split link which is not a Δ-split link.*

2 Proof of Theorem.

To prove Theorem, we prepare some lemmas.

The following is proved in [4].

Lemma 1. *For two 2-component links $\ell = k_1 \cup k_2, L = K_1 \cup K_2, \ell$ and L are self Δ-equivalent if and only if, the linking number, $Link(k_1, k_2) = Link(K_1, K_2)$ and $a_3(\ell) - a_1(\ell)(a_2(k_1) + a_2(k_2)) = a_3(L) - a_1(L)(a_2(K_1) + a_2(K_2))$, where a_i means the i-th coefficient of Conway polynomial.*

Let $V_i, V_i'(i = 1, 2)$ be solid tori in R^3 such that $V_1 \cap V_2 = V_1' \cap V_2' = \emptyset$. Denote $V_1 \cup V_2, V_1' \cup V_2'$ by $\mathcal{V}, \mathcal{V}'$ and $c_1 \cup c_2, c_1' \cup c_2'$ by Γ, Γ' respectively for the core c_i of V_i, namely for a *preferred* longitude λ_i of ∂V_i (λ_i bounds an orientable surface in $cl(R^3 - V_i)$), $f_i(\lambda_i)$ is a preferred longitude of $\partial V_i'$. For a faithful homeomorphism f_i and a knot k_i in V_i and $\ell = k_1 \cup k_2$, we denote $f_i(k_i), f_1(k_1) \cup f_2(k_2)$ by k_i', ℓ' respectively and we say that ℓ and ℓ' are *the links of a faithful pair.*

Let us denote the algebraic intersection number of k_i and a meridian disk of V_i by p_i.

Lemma 2. *Let $\ell = k_1 \cup k_2, \Gamma = c_1 \cup c_2$ and p_i be those of the above. If $Link(c_1, c_2) = 0$ and $p_1 p_2 \neq 0$, then $a_3(\ell) = p_1^2 p_2^2 a_3(\Gamma)$.*

Proof. Let λ_i be a preferred longitude of $\partial V_i, i = 1, 2$. Since $Link(c_1, c_2) = 0$, there is a non-singular orientable surface F_i in $cl(R^3 - V_1 - V_2)$ with $\partial F_i = \lambda_i$. By moving in parallel with F_i, we obtain mutually disjoint non-singular orientable surfaces $F_{i1}, ..., F_{ip_i}$ in $cl(R^3 - V_1 - V_2)$ with $\partial F_{ij} \subset \partial V_i$. Since k_i is homologous to $p_i\lambda_i$ in V_i, there is a non-singular orientable surface F_{i0} in V_i with $\partial F_{i0} = k_i \cup (-p_i\lambda_i)$. Then $\mathcal{F}_i (= F_{i0} \cup F_{i1} \cup ... \cup F_{ip_i})$ is a non-singular orientable surface in $R^3 - (\ell - k_i)$ with $\partial \mathcal{F}_i = k_i$.

Now we calculate the Sato-Levine invariant $\beta(\ell)$,([6]).

$$\begin{aligned} \beta(\ell) &= Link(\mathcal{F}_1 \cap \mathcal{F}_2, (\mathcal{F}_1 \cap \mathcal{F}_2)^+) \\ &= p_1^2 p_2^2 \, Link(F_1 \cap F_2, (F_1 \cap F_2)^+) \\ &= p_1^2 p_2^2 \, \beta(\Gamma) \end{aligned}$$

, where $(F_1 \cap F_2)^+$ means the lift of $F_1 \cap F_2$ above $F_1 \cup F_2$. Since $a_3(\ell) = -\beta(\ell)$ by [1], we obtain Lemma 2.

Lemma 3. *Let $\ell = k_1 \cup k_2, \ell' = k_1' \cup k_2'$ be the links of a faithful pair and let $\Gamma = c_1 \cup c_2, \Gamma' = c_1' \cup c_2'$ and p_i be those of the above. Suppose that $Link(c_1, c_2) = 0$ and $p_1 p_2 \neq 0$. If ℓ and ℓ' are self Δ-equivalent, then Γ and Γ' are self Δ-equivalent.*

Proof. Since ℓ and ℓ' are self Δ-equivalent, $Link(k_1', k_2') = Link(k_1, k_2) = p_1^2 p_2^2 \, Link(c_1, c_2) = 0$ and so $Link(c_1', c_2') = 0$. Moreover as $a_1(\ell) = Link(k_1, k_2)$, ([1]), we obtain that $a_1(\ell) = a_1(\ell') = 0$ and so $a_3(\ell) = a_3(\ell')$ by Lemma 1. Hence $a_3(\Gamma) = a_3(\Gamma')$ by Lemma 2. Therefore Γ and Γ' are self Δ-equivalent by Lemma 1.

Since the 3-rd coefficient of Whitehead link is 1, it is not self Δ-equivalent to the trivial link by Lemma 1. Moreover as a Δ-move is an unknotting operation of knots, ([2]), we obtain the following by Lemma 2.

Lemma 4. *Let $\ell = k_1 \cup k_2, \Gamma = c_1 \cup c_2$ and $p_i(\neq 0)$ be those of Lemma 3. If Γ is not self Δ-equivalent to $\mathcal{O}$, then ℓ is not self Δ-equivalent to $\mathcal{O}$. Especially, for the Whitehead link $\Gamma = c_1 \cup c_2, \ell = k_1 \cup c_2$ is not self Δ-equivalent to $\mathcal{O}$.*

By Lemma 3.3 in [5], we obtain the following.

Lemma 5. *Let L be an n-component link and K a component of L. Then K can be deformed into a trivial knot in $R^3 - (L - K)$ if and only if there is a claspless disk D in R^3 with $\partial D = K$ and $D \cap (L - K) = \emptyset$.*

Proof of Theorem. We easily see that the link $\ell = k_1 \cup k_2$ as illustrated in Figure 1 is an h-split link.

Fig. 1

Now let us prove that ℓ is not a Δ-split link.

Suppose that ℓ is a Δ-split link. Then there is a disjoint union $\mathcal{D} = D_1 \cup D_2$ of claspless disks D_i with $\partial \mathcal{D} = \ell, \partial D_i = k_i$ for $i = 1, 2$. Let us denote a longitude, a meridian of ∂V_1 by λ, μ respectively.

Since k_i is contained in the interior of $D_i, \Lambda_i (= D_i \cap \partial V_i)$ consists of loops for $i = 1, 2$.

Case 1. $\Lambda_1 = \emptyset$ or $\Lambda_2 = \emptyset$.

If Λ_1 is empty, $D_1 \cap \mu$ is empty and so $k_1 \cup \mu (= \partial D_1 \cup \mu)$ is self Δ-equivalent to a trivial link $\mathcal{O}$ by Lemma 5. But as $k_1 \cup \mu$ is the Whitehead link, it is not self Δ-equivalent to $\mathcal{O}$ by Lemma 1.

If Λ_2 is empty, $\lambda \cap D_2$ is empty and so $\lambda \cup k_2 (= \lambda \cup \partial D_2)$ is self Δ-equivalent to $\mathcal{O}$ by Lemma 5. But as $\lambda \cup k_2$ is the Whitehead link, we obtain a contradiction.

Therefore we assume that $\Lambda_1 \neq \emptyset$ from now on.

Case 2. There is a loop γ of Λ_1 which is not homologous to 0 on ∂V_1.

Suppose that γ is homologous to $p\lambda + q\mu$ on ∂V_1 for $p \neq 0$ or $q \neq 0$.

Case 2(a). $p \neq 0$.

Since $D_1 \cap D_2 = \emptyset$, there is a regular neighborhood, $N(\gamma : \partial V_1)$, of γ on ∂V_1 such that $N(\gamma : \partial V_1) \cap D_2 = \emptyset$. Then we may choose a simple loop γ_0

on $N(\gamma : \partial V_1)$ such that γ_0 is homologous to $p'\lambda + q'\mu$ on ∂V_1 for $p' \neq 0$. Since $\gamma_0 \cap D_2$ is empty for a claspless disk $D_2, \gamma_0 \cup k_2 (= \gamma_0 \cup \partial D_2)$ is self Δ-equivalent to $\mathcal{O}$ by Lemma 5. But as $\gamma_0 \cup k_2$ is not self Δ-equivalent to $\mathcal{O}$ by Lemma 4.

Case 2(b). $p = 0$.

In this case, γ is homologous to $q\mu$ on ∂V_1 for $q \neq 0$. Hence there is a simple loop γ_0 on $N(\gamma : \partial V_1)$ homologous to μ on ∂V_1.

If there is a simple loop ρ on ∂V_1 such that ρ is homologous to $m\lambda + n\mu$ on ∂V_1 for $m \neq 0$ and $\rho \cap D_2 = \rho \cap \Lambda_2 = \emptyset, \rho \cup k_2 = (\rho \cap \partial D_2)$ is self Δ-equivalent to $\mathcal{O}$ by Lemma 5. But as $\rho \cup k_2$ is not self Δ-equivalent to $\mathcal{O}$ by Lemma 4. Hence there is not a loop ρ satisfying the above. Moreover as $\gamma_0 \cap D_2 = \emptyset$, there are loops $\delta_1, ..., \delta_s$ of $\Lambda_2 (= D_2 \cap \partial V_1)$ such that $N(\delta_1 \cup ... \cup \delta_s : \partial V_1)$contain a simple loop δ homologous to μ on ∂V_1 and $D_1 \cap N(\delta_1 \cup ... \cup \delta_s : \partial V_1) = \emptyset$. Since $D_1 \cap \delta$ is empty, $k_1 \cup \delta (= \partial D_1 \cup \delta)$ is self Δ-equivalent to $\mathcal{O}$ by Lemma 5. But as $k_1 \cup \delta$ is the Whitehead link, it is not self Δ-equivalent to $\mathcal{O}$.

Case 3. For any loop γ of Λ_1, γ is homologous to 0 on ∂V_1.

Suppose that there are loops $\gamma_1, ..., \gamma_t$ of Λ_1 such that $N(\gamma_1 \cup ... \cup \gamma_t : \partial V_1)$ contains a simple loop κ homologous to $p\lambda + q\mu$ on ∂V_1 for $p \neq 0$. Since $N(\gamma_1 \cup ... \cup \gamma_t : \partial V_1) \cap D_2 = \emptyset, \kappa \cap D_2$ is empty and so $\kappa \cup k_2 (= \kappa \cup \partial D_2)$ is self Δ-equivalent to $\mathcal{O}$ by Lemma 5. But as $\kappa \cup k_2$ is not self Δ-equivalent to $\mathcal{O}$ by Lemma 4.

Next we consider the case that Λ_1 does not contain such loops. In this case, there is a meridian μ_1 on ∂V_1 such that $\mu_1 \cap D_1 = \mu_1 \cap \Lambda_1 = \emptyset$. Then $k_1 \cup \mu_1 (= \partial D_1 \cup \mu_1)$ is self Δ-equivalent to $\mathcal{O}$. But as $k_1 \cup \mu_1$ is the Whitehead link, it is a contradiction.

Since a full-twist can be removed by a Δ-move, Figure 2, the link L_r as illustrated in Figure 3 is self Δ-equivalent to the Whitehead link and so L_r is not self Δ-equivalent to $\mathcal{O}$.

Fig. 2

Fig. 3

Therefore we see that the link $\mathcal{L}_{p,q}$ as illustrated in Figure 4 is an h-split link but not a Δ-split link by the same discussion as above.

Fig. 4

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Department of Mathematics
Osaka Institute of Technology
Asahi, Osaka 535-8585, Japan
E-mail: shibuya@ge.oit.ac.jp

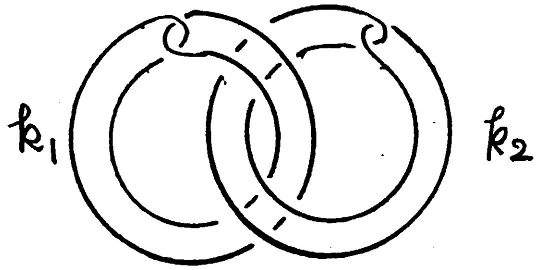

Fig. 1

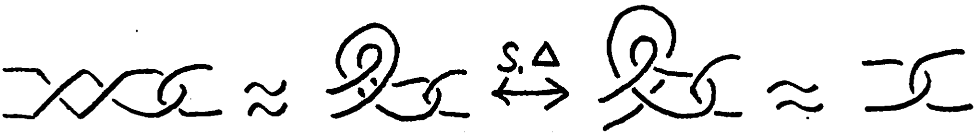

Fig. 2

Fig. 3

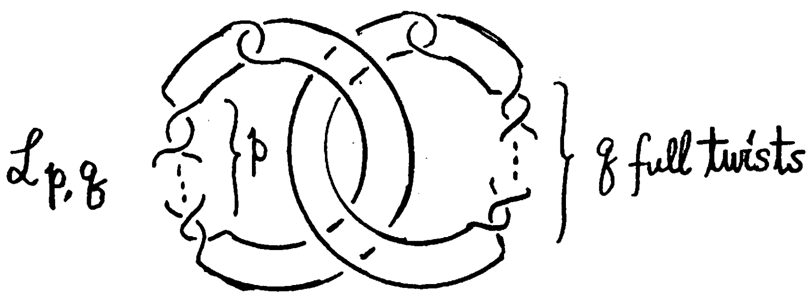

Fig. 4