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ON THE TWISTED ALEXANDER POLYNOMIALS OF KNOTS

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1. Introduction

Let K be a knot in S^3 and $G(K)$ the group of K. Given a linear representation $\rho : G(K) \longrightarrow GL(n, D)$, we can define the twisted Alexander polynomial $\widetilde{\Delta}_{\rho,K}(t)$ of K associated to ρ, where D is a unique factorization domain. If ρ is a trivial representation, then $\widetilde{\Delta}_{\rho,K}(t) = [\Delta_K(t)/(t-1)]^n$, where $\Delta_K(t)$ is the Alexander polynomial of K, and hence, $\widetilde{\Delta}_{\rho,K}(t)$ is a generalization of the Alexander polynomial. Two polynomials $\Delta_K(t)$ and $\widetilde{\Delta}_{\rho,K}(t)$ have many properties in common. For example, if K is a fibred knot, both polynomials are monic (See [GKM]). However, if the coefficient domain D is the complex number field $\mathbb{C}$, $\widetilde{\Delta}_{\rho,K}(t)$ is a polynomial (or a rational function) with complex coefficients. Very recently, however, if D is a (finite) extension field over $\mathbb{Q}$, D.Silver and S.Williams introduced the polynomial $D_{\rho,K}(t)$ over $\mathbb{Q}$ called the total twisted Alexander polynomial (associated to ρ). For the definition, see (2.1). Although the degree of $D_{\rho,K}(t)$ is much larger than that of $\widetilde{\Delta}_{\rho,K}(t), D_{\rho,K}(t)$ may have many advantages over $\widetilde{\Delta}_{\rho,K}(t)$.

In this report, we discuss these polynomials associated to two representations, abelian representations $G(K) \longrightarrow \mathbb{C}$ and parabolic representations $G(K) \longrightarrow SL(2, \mathbb{C})$. However, since the details of the main theorem on parabolic representation will appear elsewhere, we just sketch the outline of the proof of the main theorem.

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2. Preliminaries

Let $\widetilde{\Delta}_{\rho,K}(t)$ be the twisted Alexander polynomial of K associated to a linear representation $\rho : G(K) \longrightarrow GL(n,\mathbb{C})$. (For definition, see [L], [KL] or [W]). Suppose coefficients of $\widetilde{\Delta}_{\rho,K}(t)$ are in $\mathbb{Q}(w)$, where w is an algebraic number over $\mathbb{Q}$. To emphasize this, $\widetilde{\Delta}_{\rho,K}(t)$ is sometimes written as $\tilde{\Delta}_{\rho,K}(t|w)$. Now let $w_1, w_2, \cdots, w_r$ be all algebraic conjugates of w, say $w_1 = w$. Or equivalently, $w_1, w_2, \cdots, w_r$ are roots of the minimal polynomial of w (of degree r). Then $\prod_{j=1}^{r} \widetilde{\Delta}_{\rho,K}(t|w_j)$ is a polynomial (or a rational function) over $\mathbb{Q}$, and D.Silver and S.Williams call it the total twisted Alexander polynomial of K associated to ρ and we denote it by $D_{\rho,K}(t)$. Namely, we define

(2.1) $D_{\rho,K}(t) = \prod_{j=1}^{r} \widetilde{\Delta}_{\rho,K}(t|w_j)$.

Example 2.1. If $\widetilde{\Delta}_{\rho,K}(t) = 1 + \sqrt{-5}t + t^2$, then $D_{\rho,K}(t) = (1 + \sqrt{-5}t + t^2)(1 - \sqrt{-5}t + t^2) = 1 - 7t^2 + t^4$.

Throughout this paper, we use the following notations:
$\Delta_K(t)$: the Alexander polynomial of a knot K in S^3,
$M_q(K)$: the q-fold cyclic cover of S^3 branched along a knot K,
$\Delta_{M_q(K)}(t)$: the Alexander polynomial of the knot $\widetilde{K} \subset M_q(K)$, where $\widetilde{K}$ is the lift of K in $M_q(K)$,
$A \bigotimes B$: the tensor product of a matrix $A = |a_{i,j}|_{1\leq i,j\leq p}$ and $B = |b_{i,j}|_{1\leq i,j\leq q}$. $A \bigotimes B$ is a $pq \times pq$ matrix $|a_{i,j}B|_{1\leq i,j\leq p}$.

Part(I): Abelian Representations

3. Finite abelian representations

Since the commutator quotient group $G(K)/[G(K),G(K)]$ is infinite cyclic, an abelian representation of $G(K)$ is of degree 1. Consider an abelian representation $\rho : G(K) \longrightarrow \mathbb{C}$, given by $\rho(x) = \alpha$, where x is a meridian element of $G(K)$. Suppose α is a primitive p-th root of 1. Then $\widetilde{\Delta}_{\rho,K}(t) = \frac{\Delta_K(\alpha t)}{\alpha t - 1}$, and hence $D_{\rho,K}(t) = \prod_{j=1}^{p-1} \frac{\Delta_K(\alpha^j t)}{\alpha^j t - 1} = [\prod_{j=0}^{p-1} \frac{\Delta_K(\alpha^j t)}{\alpha^j t - 1}] \frac{t-1}{\Delta_K(t)}$. Now we can write $\prod_{j=0}^{p-1} \Delta_K(\alpha^j t) = F(t^p)$. Put $t^p = \tau$. Then it is known ([F] or [K]) that $F(\tau)$ is $\Delta_{M_p(K)}(\tau)$, the Alexander polynomial of the knot $\widetilde{K}$ in $M_p(K)$. Therefore we have:

Proposition 3.1. *Let $\rho : G(K) \longrightarrow \mathbb{C}$ be an abelian representation given by $\rho(x) = \alpha$, a primitive p-th root of 1. Then $D_{\rho,K}(t) = F(t^p)/\Delta_K(t)(1+t+\cdots+t^{p-1})$, where $F(t^p) = \Delta_{M_p(K)}(t^p)$.*

4. Infinite Abelian representations

Let α be a root of $\Delta_K(t^m) = 0$, $m \geq 1$, where $\deg \Delta_K(t) = 2n$. Consider an abelian representation $\rho : G(K) \longrightarrow \mathbb{C}$ given by $\rho(x) = \alpha$, where x is a meridian element of $G(K)$. In this section, we consider a slight extension of the total Alexander polynomial denoted by $\widehat{D}^{(m)}_{\rho,K}(t)$. Let $\alpha_1 (= \alpha), \alpha_2, \cdots, \alpha_d$ be all distinct roots of $\Delta_K(t^m)$. Then we define

(4.1) $\widehat{D}^{(m)}_{\rho,K}(t) = \prod_{j=1}^{d} \widehat{\Delta}_{\rho,K}(t|\alpha_j)$,

and we determine $\widehat{D}^{(m)}_{\rho,K}(t)$. For simplicity, we consider only the case where $\Delta_K(t)$ is irreducible and monic.

Let $w_1, w_2, \cdots, w_{2n}$ be the roots of $\Delta_K(t)$, and write $\Delta_K(t) = \prod_{j=1}^{2n}(t - w_j)$. Let α_j be such that ${\alpha_j}^m = w_j, j = 1, 2, \cdots, 2n$. If ζ is a primitive m-th root of 1, then $\zeta^\ell \alpha_j$ is also a root of $\Delta_K(t^m) = 0$. Now since $\widetilde{\Delta}_{\rho,K}(t|\alpha) = \frac{\Delta_K(\alpha t)}{\alpha t - 1}$, we have:

(4.2) $\widehat{D}^{(m)}_{\rho,K}(t) = \prod_{1\leq j\leq 2n, 0\leq \ell \leq m-1} \frac{\Delta_K(\zeta^\ell \alpha_j t)}{\zeta^\ell \alpha_j t - 1}$.

Now, the numerator $N(\widehat{D}^{(m)}_{\rho,K}(t))$ of $\widehat{D}^{(m)}_{\rho,K}(t)$ is equal to $\prod_{j,\ell} \Delta_K(\zeta^\ell \alpha_j t)$
$= \prod_{1\leq j,k\leq 2n} \prod_{\ell=0}^{m-1} (\zeta^\ell \alpha_j t - w_k) = \prod_{1\leq j,k\leq 2n} ({\alpha_j}^m t^m - w_k^m)$
$= \prod_{1\leq j,k\leq 2n} (w_j t^m - w_k^m) = \prod_{1\leq j,k\leq 2n} w_j (t^m - w_k^m w_j^{-1})$
$= \prod_{1\leq j,k\leq 2n} (t^m - w_k^m w_j^{-1})$, since $\prod_{j=1}^{2n} w_j = 1$.

Let $\Gamma(t) = \prod_{1\leq j,k\leq 2n}(t - w_k^m w_j^{-1})$ and let C be the companion matrix of $\Delta_K(t)$. Then $w_1, w_2, \cdots, w_{2n}$ are the eigenvalues of C and $w_1^m, w_2^m, \cdots, w_{2n}^m$ are the eigenvalues of C^m, and $w_k^m w_j^{-1}$, $1 \leq j, k \leq 2n$ are the eigenvalues of the tensor product $C^m \bigotimes C$, since ${w_j}^{-1}, 1 \leq j \leq 2n$ are eigenvalues of C. Therefore $\Gamma(t)$ is the characteristic polynomial of $C^m \bigotimes C$, and we have:

(4.3) (1) $\Gamma(t) = \det[tI_{4n^2} - C^m \bigotimes C]$, and
(2) $N(\widehat{D}^{(m)}_{\rho,K}(t)) = \Gamma(t^m)$.

On the other hand, the denominator of $\widehat{D}^{(m)}_{\rho,K}(t)$ is
$\prod_{1\leq j\leq 2n, 0\leq \ell\leq m-1}(\zeta^\ell \alpha_j t - 1) = \Delta_K(t^{-m}) = \Delta_K(t^m)$. Therefore, we have the following proposition:

Proposition 4.1.
$\widehat{D}^{(m)}_{\rho,K}(t) = \Gamma(t^m)/\Delta_K(t^m) = \det[t^m I_{4n^2} - C^m \bigotimes C]/\Delta_K(t^m)$.

Next, we prove some properties of $\Gamma(t^m)$. Since $\Delta_K(t)$ is symmetric, we may write $w_{n+q} = w_q, 1 \leq q \leq n$. Therefore, we see:

(4.4) $\Gamma(t)$
$= \prod_{1\leq j,k\leq n}[(t - w_k^m w_j)(t - w_k^m w_j^{-1})(t - w_k^{-m} w_j)(t - w_k^{-m} w_j^{-1})$
$= \prod_{k=1}^n[(t-w_k^{m+1})(t-w_k^{m-1})(t-w_k^{-(m-1)})(t-w_k^{-(m+1)})] \prod_{1\leq j\neq k\leq n}$
$[(t - w_k^m w_j)(t - w_k^m w_j^{-1})(t - w_k^{-m} w_j)(t - w_k^{-m} w_j^{-1})]$.

As is stated in the previous section, we see:
$\prod_{k=1}^n[(t - w_k^{m-1})(t - w_k^{-(m-1)})] = \prod_{k=1}^{2n}(t - w_k^{m-1}) = \Delta_{M_{m-1}(K)}(t)$, the Alexander polynomial of $\widetilde{K}$ in $M_{m+1}(K)$.
Similarly, $\prod_{k=1}^n[(t-w_k^{m+1})(t-w_k^{-(m+1)})] = \prod_{k=1}^{2n}(t-w_k^{m+1}) = \Delta_{M_{m+1}(K)}(t)$.
Therefore, we obtain:

Proposition 4.2. $\Gamma(t) = \Delta_{M_{m-1}(K)}(t)\Delta_{M_{m+1}(K)}(t)h(t)$, *where*
(4.5) $h(t) = \prod_{1\leq j\neq k\leq n}[(t-w_k^m w_j)(t-w_k^m w_j^{-1})(t-w_k^{-m} w_j)(t-w_k^{-m} w_j^{-1})]$,
and
(4.6) $\widehat{D}^{(m)}_{\rho,K}(t) = \Delta_{M_{m-1}(K)}(t^m)\Delta_{M_{m+1}(K)}(t^m)h(t^m)/\Delta_K(t^m)$.

When $m = 1$ or 2, we can obtain more precise formulas on $\widehat{D}^{(m)}_{\rho,K}(t)$. In fact, we have:

Proposition 4.3. *(1)* $\widehat{D}^{(1)}_{\rho,K}(t)$ *is of the form:*
$\widehat{D}^{(1)}_{\rho,K}(t) = (t-1)^{2n} f(t)g(t)^2/\Delta_k(t)$, *where* $f(t^2) = \Delta_K(t)\Delta_K(-t)$ *and* $g(t) \in \mathbb{Q}[t]$.
(2) $\widehat{D}^{(2)}_{\rho,K}(t)$ *is a polynomial, i.e.* $\widehat{D}^{(2)}_{\rho,K}(t) = \Delta_{M_3(K)}(t^2)h(t^2)$.

Proof of (1). Let $m = 1$. Since $\prod_{k=1}^{2n}(t - w_k^{m-1}) = (t-1)^{2n}$ and $\Delta_{M_2(K)}(t^2) = \Delta_K(t)\Delta_K(-t)$, it remains to show that
$h(t) = \prod_{1\leq j\neq k\leq n}[(t - w_k^m w_j)(t - w_k^m w_j^{-1})(t - w_k^{-m} w_j)(t - w_k^{-m} w_j^{-1})]$ is a square of a polynomial $g(t) \in \mathbb{Q}[t]$.

Now we notice that
$h(t) = \prod_{1\leq j<k\leq n}[(t-w_k^m w_j)^2(t-w_k^m w_j^{-1})^2(t-w_k^{-m} w_j)^2(t-w_k^{-m} w_j^{-1})^2]$,
and hence it is sufficient to prove
$g(t) = \prod_{1\leq j<k\leq n}[(t - w_k^m w_j)(t - w_k^m w_j^{-1})(t - w_k^{-m} w_j)(t - w_k^{-m} w_j^{-1})]$ is a polynomial over $\mathbb{Q}$. However it is immediate, since $g(t)$ is remained unchanged under any permutation $w_p \leftrightarrow w_q^{\pm 1}$.

Proof of (2). Since $m = 2$, $M_{m-1}(K) = S^3$ and hence $\Delta_{M_{m-1}(K)}(t) = \Delta_K(t)$, and (2) follows. □

Finally we evaluate $\widehat{D}^{(m)}_{\rho,K}(t)$ at $t^m = 1$ and $t^m = -1$. Since $\widehat{D}^{(m)}_{\rho,K}(t)$ is a rational function in t^m, we write $\widehat{D}^{(m)}_{\rho,K}(t)$ as $\delta^{(m)}_{\rho,K}(\tau)$, where $\tau = t^m$. We evaluate $\delta^{(m)}_{\rho,K}(\pm 1)$.

Proposition 4.4. *(1) If m is even, then (a) $\delta^{(m)}_{\rho,K}(1) = s^2$ and (b) $\delta^{(m)}_{\rho,K}(-1) = \Delta_K(-1)r^2$, where s and r are integers.*
(2) If $m(\geq 3)$ is odd, then (a) $\delta^{(m)}_{\rho,K}(1) = \Delta_K(-1)^2u^2$ and (b) $\delta^{(m)}_{\rho,K}(-1) = v^2/\Delta_K(-1)$, where u and v are integers.
Thus, for any m, $\delta^{(m)}_{\rho,K}(1)\delta^{(m)}_{\rho,K}(-1) = \Delta_K(-1)a^2$ for some integer a.

Proof. Consider the polynomial $\Gamma(t) = \prod_{1\leq j,k\leq 2n}(t - w_k^m w_j^{-1})$. First we show that $\Gamma(1)$ and $\Gamma(-1)$ both are squares of some integers. Consider $\Gamma(1)$. As we see in (4.4) $\Gamma(1)$ is rewritten as follows:
(4.7) $\Gamma(1)$

$$= \prod_{k=1}^n[(1-w_k^{m+1})(1-w_k^{m-1})(1-w_k^{-(m-1)})(1-w_k^{-(m+1)})]\prod_{1\leq j\neq k\leq n}$$
$$[(1 - w_k^m w_j)(1 - w_k^m w_j^{-1})(1 - w_k^{-m} w_j)(1 - w_k^{-m} w_j^{-1})].$$

In the first product, $(1-w_k^{m+1})(1-w_k^{m-1})(1-w_k^{-(m-1)})(1-w_k^{-(m+1)})$
$= (1 - w_k^{m+1})(1 - w_k^{m-1})w_k^{-(m-1)}(w_k^{(m-1)} - 1)w_k^{-(m+1)}(w_k^{m+1} - 1)$
$= w_k^{-2m}(1 - w_k^{m+1})^2(1 - w_k^{m-1})^2 = [w_k^{-m}(1 - w_k^{m+1})(1 - w_k^{m-1})]^2$
$= [(w_k^m + w_k^{-m}) - (w_k + w_k^{-1})]^2$.

Similarly, in the second product, for $j \neq k$,
$(1 - w_k^m w_j)(1 - w_k^m w_j^{-1})(1 - w_k^{-m} w_j)(1 - w_k^{-m} w_j^{-1})$
$= (1 - w_k^m w_j)w_k^m w_j^{-1}(w_k^{-m} w_j - 1)(1 - w_k^{-m} w_j)w_k^{-m} w_j^{-1}(w_k^m w_j - 1)$
$= w_j^{-2}(1 - w_k^m w_j)^2(1 - w_k^{-m} w_j)^2 = [(w_j + w_j^{-1}) - (w_k^m + w_k^{-m})]^2$.

Let $\pi(j,k) = (w_j + w_j^{-1}) - (w_k^m + w_k^{-m})$. Then the above computation shows $\Gamma(1) = \prod_{1\leq j,k\leq n}\pi(j,k)^2$. Obviously, $\prod_{1\leq j,k\leq n}\pi(j,k)$ is remained unchanged under any permutation $w_p \leftrightarrow w_q^{\pm 1}$, and hence $\prod_{1\leq j,k\leq n}\pi(j,k) \in \mathbb{Q}$. Therefore $\Gamma(1)$ is a square of an integer.

A similar argument works for $\Gamma(-1)$.

Now, we return to the proof of Proposition 4.4. (1) Suppose $m = 2q$. Then $\delta^{(m)}_{\rho,K}(1) = \Gamma(1)/\Delta_K(1) = s^2$ for some integer s. Also, $\delta^{(m)}_{\rho,K}(-1) = \Gamma(-1)/\Delta_K(-1)$ and $\Gamma(-1) = \Delta_{M_{m-1}(K)}(-1)\Delta_{M_{m+1}(K)}(-1)h(-1)$. Now we see

$\Delta_{M_{m-1}(K)}(-1) = \prod_{k=1}^{2n}(1+w_k^{m-1}) = \prod_{k=1}^{2n}(1+w_k)\prod_{k=1}^{2n}(1-w_k+w_k^2-\cdots+w_k^{2q-2}) = \Delta_K(-1)\prod_{k=1}^{2n}(1-w_k+w_k^2-\cdots+w_k^{2q-2})$.

Similarly, $\Delta_{M_{m+1}(K)}(-1) = \Delta_K(-1)\prod_{k=1}^{2n}(1-w_k+w_k^2-\cdots+w_k^{2q})$.

Since $\Gamma(-1)$ is a square of an integer, we see $\Gamma(-1) = [\Delta_K(-1)]^2 r^2$ for some integer r. This proves (1).

(2) Suppose $m = 2q+1, q \geq 1$. Since $m \pm 1$ are even, $\Delta_{M_{m-1}(K)}(1)$ is the order of $H_1(M_{m-1}(K);\mathbb{Z})$ that is $\Delta_K(-1)u$ for some integer u. Similarly, $\Delta_{M_{m+1}(K)}(1) = \Delta_K(-1)v$ for some integer v. Further, $\Gamma(-1)$ is a square of an integer. Therefore (2) (a), (b) follows immediately. □

5. Examples

Let α be such that $\Delta_K(\alpha^m) = 0$. Let $\rho : G(K) \longrightarrow \mathbb{C}$, $\rho(x) = \alpha$ be an abelian representation. We use the notation introduced in the previous section.

(1) Let $\Delta_K(t) = 1-t+t^2$. Then
$\widehat{D}^{(1)}_{\rho,K}(t) = (t-1)^2(1+t+t^2)/\Delta_K(t)$, $\widehat{D}^{(2)}_{\rho,K}(t) = (1+t^2)^2$, $\widehat{D}^{(3)}_{\rho,K}(t) = (1+t^3+t^6)^2/\Delta_K(t^3)$.
Further, $\delta^{(2)}_{\rho,K}(1) = 2^2$, and $\delta^{(2)}_{\rho,K}(-1) = 0$. Also, $\delta^{(3)}_{\rho,K}(1) = 3^2$, and $\delta^{(3)}_{\rho,K}(-1) = 1/3$.

(2) Let $\Delta_K(t) = 1-3t+t^2$. Then
$\widehat{D}^{(1)}_{\rho,K}(t) = (t-1)^2(1-7t+t^2)/\Delta_K(t)$, $\widehat{D}^{(2)}_{\rho,K}(t) = 1-18t^2+t^4$, $\widehat{D}^{(3)}_{\rho,K}(t) = (1-7t^3+t^6)(1-47t^3+t^6)/\Delta_K(t^3)$.
Further, $\delta^{(2)}_{\rho,K}(1) = 2^4$, and $\delta^{(2)}_{\rho,K}(-1) = 5.2^2$. Also, $\delta^{(3)}_{\rho,K}(1) = 3^2.5^2$, and $\delta^{(3)}_{\rho,K}(-1) = 3^2.7^2/5$.

(3) Let $\Delta_K(t) = 1-3t+5t^2-3t^3+t^4$. Then
$\widehat{D}^{(1)}_{\rho,K}(t) = (t-1)^4(1+t+9t^2+t^3+t^4)(1-3t+t^2-3t^3+t^4)^2/\Delta_K(t)$,
$\widehat{D}^{(2)}_{\rho,K}(t) = (1+9t^2+29t^4+9t^6+t^8)(1-3t^2+7t^4+63t^6+120t^8+63t^{10}+7t^{12}-3t^{14}+t^{16})$,
$\widehat{D}^{(3)}_{\rho,K}(t) = (1+t^3+9t^6+t^9+t^{12})(1+17t^3+81t^6+17t^9+t^{12})(1+9t^3+107t^6+531t^9+1008t^{12}+531t^{15}+107t^{18}+9t^{21}+t^{24})/\Delta_K(t^3)$.
Further, $\delta^{(2)}_{\rho,K}(1) = 2^8.7^2$, and $\delta^{(2)}_{\rho,K}(-1) = 13.2^4$. Also, $\delta^{(3)}_{\rho,K}(1) = 2^8.3^4.13^2$, and $\delta^{(3)}_{\rho,K}(-1) = 2^4.3^4.7^2/13$.

In the last example below, we consider the case where $\Delta_K(t)$ is not monic.

(4) Let $\Delta_K(t) = 3 - 7t + 9t^2 - 7t^3 + 3t^4$. Then
$\widehat{D}^{(1)}_{\rho,K}(t) = (1/9)(t-1)^4(9+5t+t^2+5t^3+9t^4)(9-9t+13t^2-9t^3+9t^4)^2/(1/3)\Delta_K(t)$,
$\widehat{D}^{(2)}_{\rho,K}(t) = (1/9)(27+35t^2+45t^4+35t^6+27t^8)(729+1701t^2+1197t^4-1029t^6-2492t^8-1029t^{10}+1197t^{12}+1701t^{14}+729t^{16})/(1/3)$,
$\widehat{D}^{(3)}_{\rho,K}(t) = (1/9)(9+5t^3+t^6+5t^9+9t^{12})(81-7t^3+113t^6-7t^9+81t^{12})(6561+16767t^3+9567t^6-14715t^9-27896t^{12}-14715t^{15}+9567t^{18}+16767t^{21}+6561t^{24})/(1/3)\Delta_K(t^3)$.
Further, $\delta^{(2)}_{\rho,K}(1) = 2^4.13^4/3$, and $\delta^{(2)}_{\rho,K}(-1) = 2^4.29/3$. Also, $\delta^{(3)}_{\rho,K}(1) = 2^4.3^2.23^2.29^2/3$, and $\delta^{(3)}_{\rho,K}(-1) = 2^8.3^2.17^2/3.29$.

Part (II): Parabolic Representations

6. Definitions

In this part, we consider exclusively 2-bridge knots. Let $K(r)$, $r > 0$ or $K(\alpha,\beta)$, be a 2-bridge knot of type (α,β), where $0 < \beta < \alpha$, $0 < r = \beta/\alpha < 1$ and $\gcd(\alpha,\beta) = 1$. Also, we assume that both α and β are odd.

Now, a Wirtinger presentation of the group of $K(r)$ is of the form:
(6.1) $G(K) = \langle x, y | WxW^{-1}y^{-1} = 1\rangle$.

A parabolic representation of $G(K(r))$, $\rho : G(K(r)) \longrightarrow SL(2,\mathbb{C})$ will be found as follows.
Set $\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\rho(y) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. For convenience, we denote by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. To determine z, we compute $\rho(W) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$ and let $s_0 \neq 0$ be a zero of $a(z)$. Then
(6.2) $\rho(x) = A$ and $\rho(y) = B(s_0)$
give a parabolic representation of $G(K(r))$ into $SL(2,\mathbb{C})$, [R]. We call ρ the canonical parabolic representation of $G(K(r))$ and $a(z)$ the representation polynomial of ρ. It is known [R] that
(6.3) (1) $a(z)$ is an integer polynomial with the leading coefficient 1, and hence s_0 is an algebraic integer.

(2) $a(z)$ is separable and $\deg a(z) = \frac{\alpha-1}{2}$, where $r = \beta/\alpha$.

Now let $\widetilde{\Delta}_{\rho,K(r)}(t)$ be the twisted Alexander polynomial of $K(r)$ associated to ρ given by (6.2).

Our main theorem in this part is concerned about $\widetilde{\Delta}_{\rho,K(r)}(t)$.

7. Main Theorem

Let $H(p)$ be the set of all 2-bridge knots $K(r)$ such that there is an epimorphism from $G(K(r))$ to $G(K(1/p))$, the group of a torus knot of type $(p, 2)$.

Now, the following proposition is well-known [GR], [ORS]:

Proposition 7.1. *(1) $K(r) \in H(p)$ if and only if $G(K(r))$ is mapped onto a (non-trivial) free product $\mathbb{Z}/2 * \mathbb{Z}/p$.*
(2) $K(r) \in H(p)$ if and only if the continued fraction of $r = \frac{\beta}{\alpha}$ is of the form, where k_i and m_j are non-zero integers.

$$r = \frac{\beta}{\alpha} = \cfrac{1}{pk_1 - \cfrac{1}{2m_1 - \cfrac{1}{pk_2 - \ddots - \cfrac{1}{2m_q - \cfrac{1}{pk_{q+1}}}}}}$$

We denote the continued fraction above by the symbol
$r = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_q, pk_{q+1}]$.

Using representation polynomials, we can give another criterion whether or not $K(r)$ is in $H(p)$.

From now on, we always assume that $p = 2n + 1$ and $G(K(1/p))$ has the following Wirtinger presentation :
(7.1) $G(K(1/p)) = \langle x, y | R_0 = 1\rangle$, $R_0 = W_0 x W_0{}^{-1} y^{-1}$, $W_0 = (xy)^n$.

Let ρ_0 be the canonical parabolic representation of $G(K(1/p))$ to $SL(2, \mathbb{C})$ and let $a_n(z)$ be the representation polynomial of ρ_0. By definition, $a_n(z)$ is the $(1, 1)$-entry of the matrix $\rho_0(xy)^n$. Then, we have:

Proposition 7.2. *$K(r) \in H(p)$ if and only if the parabolic representation polynomial $a(z)$ of $\rho : G(K(r)) \longrightarrow SL(2, \mathbb{C})$ is divisible by $a_n(z)$.*

The polynomial $a_n(z)$ is well-studied and we list some of its properties [S].
(7.2) (1) $a_n(z) = \prod_{q|p} \chi_q(z)$, where the product runs over all integers q dividing p, and $\chi_q(z)$ is an irreducible, separable integer polynomial with the leading coefficient 1.
(2) $a_n(z) = \sum_{k=0}^{n} \binom{n+k}{2k} z^k$.
(3) The degree of $\chi_q(z)$ is $\phi(q)/2$, where $\phi(q)$ is Euler function, i.e. the

number of $j, 1 \leq j \leq q$, such that $\gcd(j,q) = 1$.
We note that $\chi_1(z) = 1$, and usually we omit it.

Example.7.3. (1) $a_1(z) = \chi_3(z) = 1 + z$.
(2) $a_2(z) = \chi_5(z) = 1 + 3z + z^2$.
(3) $a_3(z) = \chi_7(z) = 1 + 6z + 5z^2 + z^3$.
(4) $\chi_9(z) = 1 + 9z + 6z^3 + z^4$.

Now let $K(r)$ be an element of $H(p)$. Then there is an epimorphism $\varphi : G(K(r)) \longrightarrow G(K(1/p))$ such that $\varphi(x) = x$ and $\varphi(y) = y$. Using a zero s_p of $a_n(z)$, we can write $\rho_0(x) = A$ and $\rho_0(y) = B(s_p)$. Combining φ and ρ_0, we have a parabolic representation ρ of $G(K(r))$ as $\rho = \rho_0\varphi : G(K(r)) \longrightarrow G(K(1/p)) \longrightarrow L(2,\mathbb{C})$. Let $\widetilde{\Delta}_{\rho_0,K(1/p)}(t)$ and $\widetilde{\Delta}_{\rho,K(r)}(t)$ be the twisted Alexander polynomials of $K(1/p)$ and $K(r)$, respectively. Then it is easy to see that
(7.3) (1) $\widetilde{\Delta}_{\rho_0,K(1/p)}(t)$ and $\widetilde{\Delta}_{\rho,K(r)}(t)$ are polynomials over $\mathbb{Z}[s_p]$, and
(2) $\widetilde{\Delta}_{\rho_0,K(1/p)}(t)$ divides $\widetilde{\Delta}_{\rho,K(r)}(t)$.
Therefore, we can write
(7.4) $\widetilde{\Delta}_{\rho,K(r)}(t) = \lambda_{\rho,K(r)}(t)\widetilde{\Delta}_{\rho_0,K(1/p)}(t)$, where $\lambda_{\rho,K(r)}(t) \in \mathbb{Z}[s_p][t]$.

The main theorem of Part II is the following:

Theorem 7.4 [HM Theorem A].
(1) $\lambda_{\rho,K(r)}(1) = 1$, *and*
(2) $\lambda_{\rho,K(r)}(-1) = \mu^2, \mu \in \mathbb{Z}[s_p]$.

To illustrate Theorem 7.4, we give two examples.

Example 7.5. (1) Let $r = 19/45$. Since $19/45 = [3,2,3,2,3]$, we see $K(19/45) \in H(3)$.
Then $\widetilde{\Delta}_{\rho,K(r)}(t) = (1+t^2)(25-72t+95t^2-72t^3+25t^4)$. Since $\widetilde{\Delta}_{\rho_0,K(1/3)}(t) = 1 + t^2$, it follows that $\lambda_{\rho,K(r)}(1) = 1$ and $\lambda_{\rho,K(r)}(-1) = 289 = 17^2$.
(2) Let $r = 19/85$. Since $19/85 = [5,2,10]$, we see $K(19/85) \in H(5)$ and $\lambda_{\rho,K(r)}(1) = 1$ and $\lambda_{\rho,K(r)}(-1) = (8s_5 + 15)^2$, where s_5 is a zero of $a_2(z) = \chi_5(z) = 1 + 3z + z^2$.

8. Outline of the proof of Theorem 7.4.

Suppose $K(r) \in H(p)$ and let $X = A$ and $Y = B(s_p)$. First, we define a free algebra $A(x,y : \mathbb{Z}[s_p])$ over $\mathbb{Z}[s_p]$ constructed from the free group $F(x,y)$. Let $f : A(x,y : \mathbb{Z}[s_p]) \longrightarrow M_{2,2}(\mathbb{Z}[s_p])$ be an (algebra) homomorphism defined by $f(x) = X$ and $f(y) = Y$, where $M_{2,2}(\mathbb{Z}[s_p])$ is the ring of 2×2 matrices over $Z[s_p]$.

Let $S(x,y) = f^{-1}(0)$ be the kernel of f. Then $\widetilde{A}(s_p) = A(x,y : \mathbb{Z}[s_p])/S(x,y)$ is a non-commutative $\mathbb{Z}[s_p]$-algebra. For example, $(x-1)^2 \in S(x,y)$, since $(X-I)^2 = 0$. Also, $(xy)^n x(xy)^{-n}y^{-1} - 1 \in S(x,y)$.

Now, consider Wirtinger presentations of $G(K(r))$ and $G(K(1/p))$ given by (6.1) and (7.1). Since $K(r) \in H(p)$, the homomorphism $\varphi : G(K(r)) \longrightarrow G(K(1/p))$ given by $\varphi(x) = x$ and $\varphi(y) = y$ sends R to 1 in $G(K(1/p))$. Therefore, we can write

(8.1) $R = \prod_{j=1}^{m} u_j {R_0}^{\varepsilon_j} {u_j}^{-1}$, where $u_j \in F(x,y)$ and $\varepsilon_j = \pm 1$.

Let $\Phi : \widetilde{A}(s_p) \longrightarrow M_{2,2}(\mathbb{Z}[s_p][t])$ be a ring homomorphism given by $\Phi(x) = Xt$ and $\Phi(y) = Yt$. Then $\frac{\partial R}{\partial x} = [\sum_{j=1}^{m} \varepsilon_j u_j](\frac{\partial R_0}{\partial x})$, where $\frac{\partial}{\partial x}$ denotes Fox free derivative, and hence, by definition, $\widetilde{\Delta}_{\rho,K(r)}(t)$ $= \det(\sum_{j=1}^{m} \varepsilon_j u_j)^{\Phi} \widetilde{\Delta}_{\rho_0,K(1/p)}(t)$ and $\lambda_{\rho,K(r)}(t) = \det(\sum_{j=1}^{m} \varepsilon_j u_j)^{\Phi}$. Therefore, we must show

(8.2) (1) $\lambda_{\rho,K(r)}(1) = \det(\sum_{j=1}^{m} \varepsilon_j u_j)^{\Phi_0} = 1$, and

(2) $\lambda_{\rho,K(r)}(-1) = \det(\sum_{j=1}^{m} (-1)^{\ell(u_j)} \varepsilon_j u_j)^{\Phi_0} = \mu^2$, where $\Phi_0 = \Phi|_{t=I}$ and $\ell(u_j)$ is the length of u_j.

However, (8.2) follows immediately from Proposition 8.1 below.

Proposition 8.1[HM, Proposition 5.1]. *(1) The element* $\lambda(r) = \sum_{j=1}^{m} \varepsilon_j u_j$ *in* $\widetilde{A}(s_p)$ *is a single element, i.e.* $\lambda(r) = \pm w$ *for some element* w *in* $F(x,y)$.

(2) The element $\widetilde{\lambda}(r) = \sum_{j=1}^{m} (-1)^{\ell(u_j)} \varepsilon_j u_j$ *is a constant multiple of a single element, i.e.* $\widetilde{\lambda}(r) = \mu w$ *for some* $\mu \in \mathbb{Z}[s_p]$ *and* $w \in F(x,y)$.

In fact, we prove that the element w is either y or $(yx)^{n+1}$. The number μ is a knot invariant that is recursively determined in the next section.

The proof of Proposition 8.1 is done by induction on the length of the continued fraction of r. Induction argument, however, is simplified considerably by the following one of the key propositions.

Proposition 8.2 [HM, Propositions 6.3 and 8.1]. *Let* $r = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_q, pk_{q+1}]$ *and* $r' = [pk_1', 2m_1, pk_2', 2m_2, \cdots, 2m_q, pk_{q+1}']$ *be continued fractions of* r *and* r'. *If, for some* j, $k_j \equiv k_j'$ (mod *4), then* $\lambda(r) = \lambda(r')$ *and* $\widetilde{\lambda}(r) = \widetilde{\lambda}(r')$.

9. Evaluation of μ.

Suppose $K(r) \in H(p)$.Let$r$= $[pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_q, pk_{q+1}]$ be a continued fraction of r. We define two rational numbers r' and $\widetilde{r}$ in terms of other continued fractions:
$r' = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_{q-1}, pk_q]$ and
$\widetilde{r} = [pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_{q-1}, p(k_q + k_{q+1})]$.
By Proposition 8.2, we may assume $1 \leq k_j \leq 3$, $1 \leq j \leq q+1$.

Now if $k_q + k_{q+1} \equiv 0 \pmod 4$, $\widetilde{r}$ is reduced to $[pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_{q-2}, pk_{q-1}]$.
Further, $[pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_j, 0, 2m_{j+1}, pk_{j+2}, \cdots, 2m_q, pk_{q+1}]$ is reduced to
$[pk_1, 2m_1, pk_2, 2m_2, \cdots, 2(m_j + m_{j+1}), pk_{j+2}, \cdots, 2m_q, pk_{q+1}]$.
Also, $[pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_{j-1}, pk_j, 0, pk_{j+1}, 2m_{j+1}, \cdots, 2m_q, pk_{q+1}]$ is reduced to
$[pk_1, 2m_1, pk_2, 2m_2, \cdots, 2m_{j-1}, p(k_j + k_{j+1}), 2m_{j+1}, \cdots, 2m_q, pk_{q+1}]$.

Given three continued fractions r, r' and $\widetilde{r}$, we have the following recursion formula for μ. Denote by $\mu(r)$ the invariant μ for $K(r)$ appeared in (8.2)(2). Then we have:
(9.1) $\mu(r) = \nu\mu(r') + \mu(\widetilde{r})$.

Here, (1) $\mu[0] = 0, \mu[p] = \mu[2p] = \mu[3p] = -1$, and (2) ν is of the form: $\nu = m_q b_n \sigma(k_{q+1}, M)$, where b_n is the (1,2)-entry of $\rho(xy)^n$, $M \equiv \sum_{j=1}^{q+1} k_j \pmod 4$ and $\sigma(k_{q+1}, M)$ is given by the table below:
$\sigma(1,0) = 4$, $\sigma(1,1) = -8$, $\sigma(1,2) = -4$, $\sigma(1,3) = -8$, $\sigma(2,0) = 8$, $\sigma(2,1) = 8$, $\sigma(2,2) = -8$, $\sigma(2,3) = -8$, $\sigma(3,0) = 4$, $\sigma(3,1) = 8$, $\sigma(3,2) = 4$, $\sigma(3,3) = -8$.

Example 9.1. Let $r = 19/45 = [3,2,3,2,3] \in H(3)$. Since $p = 3, n = 1$ and $b_1 = 1$, it follows $\mu(r) = \nu_1\mu[3,2,3] + \mu[3,2,6] = \nu_1(\nu_2\mu[3] + \mu[6]) + \nu_3\mu[3] + \mu[9] = -\nu_1\nu_2 - \nu_1 - \nu_3 - 1$.

Now $\nu_1 = 1.1.\ \sigma(1,3) = -8$, $\nu_2 = 1.1.\ \sigma(1,2) = -4$, $\nu_3 = 1.1.\ \sigma(2,3) = -8$, and hence $\mu(r) = -32 + 8 + 8 - 1 = -17$.

(2) Let $r = 19/85 = [5,2,10] \in H(5)$. Since $p = 5, n = 2$ and $b_2 = 2 + s_5$, s_5 being a zero of $a_5(z) = \chi_5(z)$, we see that $\mu(r) = \nu\mu[5] + \mu[15] = -\nu - 1$, and $\nu = 1.(2 + s_5)\sigma(2,3) = -8(2 + s_5)$, and hence, $\mu(r) = 8(2 + s_5) - 1 = 8s_5 + 15$. (See Example 7.5.)

10. Silver-Williams Conjecture

In this last section, we study the total twisted Alexander associated to the canonical parabolic representation ρ of a 2-bridge knot

$K(r), r = \beta/\alpha$. Let s_0 be a zero of $a(z)$, the representation polynomial of ρ. Thus ρ is given by $\rho(x) = A$ and $\rho(y) = B(s_0)$. Let $D_{\rho,K(r)}(t)$ be the total Alexander polynomial of $K(r)$ associated to ρ. Then D.Silver and S.Williams propose the following conjecture:

Conjecture 10.1. *(1) $D_{\rho,K(r)}(1) = 2^\ell$ and (2) $D_{\rho,K(r)}(-1) = m^2 2^\ell$, where ℓ is the degree of the minimal polynomial of s_0 and m is an integer.*

In this section, first we prove Conjecture 10.1 for torus knots $K(1/p)$, and then we prove the conjecture partially for a knot $K(r)$ in $H(p)$.

Now to prove Conjecture 10.1, we need to specify a root s_0. Suppose s_0 is a zero of $\chi_q(z)$, say $s_0 = \gamma_q$, where q is a divisor of p. Since $\chi_q(z)$ is irreducible, it is the minimal polynomial of γ_q. Let $\xi_q : G(K(1/p)) \longrightarrow SL(2,\mathbb{C})$ be the canonical parabolic representation of $G(K(1/p))$ given by $\xi_q(x) = A$ and $\xi_q(y) = B(\gamma_q)$. Since γ_q is a zero of $\chi_q(z)$, ξ_q also defines the canonical parabolic representation of $G(K(1/q))$ into $SL(2,\mathbb{C})$.

On the other hand, since γ_q is a zero of $\chi_q(z)$, it is known [R] that there is an epimorphism $\psi : G(K(1/p)) \longrightarrow G(K(1/q))$. Further, ξ_q induces the canonical parabolic representation $\widetilde{\xi}_q : G(K(1/q)) \longrightarrow SL(2,\mathbb{C})$ given by $\widetilde{\xi}_q(x) = A$ and $\widetilde{\xi}_q(y) = B(\gamma_q)$. Combining ψ and $\widetilde{\xi}_q$, we recover ξ_q as $\xi_q = \widetilde{\xi}_q\psi$.

Now we have a couple of relationships between the total twisted Alexander polynomial $D_{\xi_q,K(1/p)}(t)$ and $D_{\widetilde{\xi}_q,K(1/q)}(t)$.

Proposition 10.2 [HM, Proposition 3.5 and Theorem 10.2]. *(1) $D_{\xi_q,K(1/p)}(t) = (1-t^{2q}+t^{4q}-\cdots+t^{2(d-1)q})^{\ell_q} D_{\widetilde{\xi}_q,K(1/q)}(t)$, where $d = p/q$ and $\ell_q = \deg \chi_q(z)$.*
(2) Let γ_u be a zero of $\chi_u(z)$, $u \mid p$. Then we have $\prod_{u|p} D_{\xi_u,K(1/p)}(t) = (1+t^2)(1+t^{4n+2})^{n-1}$, where the product runs over all canonical parabolic representations $\xi_u : G(K(1/p)) \longrightarrow SL(2,\mathbb{C})$.
(3) $D_{\xi_q,K(1/p)}(\pm 1) = 2^{\ell_q}$.

In particular, if p is a prime ≥ 3, then $a_n(z) = \chi_p(z)$ and hence Proposition 10.2(2) shows
(10.1) $D_{\xi_p,K(1/p)}(t) = (1+t^2)(1+t^{4n+2})^{n-1}$.
Therefore, $D_{\xi_p,K(1/p)}(\pm 1) = 2^n$. Note $n = (p-1)/2 = \deg \chi_p(z)$, if p is a prime.

If p is not prime, then (10.1) may not be true. (See Example 10.4.)

To prove Conjecture 10.1 for $K(1/p)$ or to prove Proposition 10.2 (3), we note that $\prod_{u|p} D_{\xi_u,K(1/p)}(\pm 1) = 2^n$ and show that $\mathrm{D}_{\tilde{\xi}_q,K(1/q)}(\pm 1) = 2^{\ell_q}$.

Let $p = {p_1}^{c_1}{p_2}^{c_2}\cdots{p_k}^{c_k}$ be the factorization of p into primes. If $\sum_{j=1}^k c_j = 1$, then Proposition 10.2 (3) follows from (10.1) and then use induction on $\sum_{j=1}^k c_j$.

Finally, we prove Conjecture 10.1 partially for a knot $K(r)$ in $H(p)$. To be more precise, let φ: $G(K(r)) \longrightarrow G(K(1/p))$ be an epimorphism. As before, let γ_q be a zero of $\chi_q(z)$. Using $\xi_q : G(K(1/p)) \longrightarrow SL(2,\mathbb{C}), \xi_q(x) = A$ and $\xi_q(y) = B(\gamma_q)$, we have the canonical parabolic representation $\xi_q\varphi : G(K(r)) \longrightarrow G(K(1/p)) \longrightarrow SL(2,\mathbb{C})$. Then the following proposition partially proves Conjecture 10.1 for our case.

Proposition 10.3. *(1)* $D_{\xi_q\varphi,K(r)}(1) = 2^{\ell_q}$, *(2)* $D_{\xi_q\varphi,K(r)}(-1) = {\mu_q}^2 2^{\ell_q}$, *where* ℓ_q *is the degree of* $\chi_q(z)$ *and* μ_q *is an integer.*

Proposition 10.3 follows from Theorem 7.4 and Proposition 10.2.

Example 10.4. (1) Let $p = 9$ and $n = 4$. Then $a_4(z) = \chi_3(z)\chi_9(z)$. First, by (10.1), $D_{\xi_3,K(1/3)}(t) = 1 + t^2$, and by Proposition 10.2(1), we see
$D_{\xi_3,K(1/9)}(t) = (1+t^2)(1-t^6+t^{12})$. Further, by Proposition 10.2(2),
$D_{\xi_3,K(1/9)}(t)D_{\xi_9,K(1/9)}(t) = (1+t^2)(1+t^{18})^3$, and hence,
$D_{\xi_9,K(1/9)}(t) = (1+t^2)(1+t^{18})^3/(1+t^2)(1-t^6+t^{12}) = (1+t^{18})^2(1+t^6)$.
Therefore, $D_{\xi_3,K(1/9)}(\pm 1) = 2$ and $D_{\xi_9,K(1/9)}(\pm 1) = 2^3$. Note that deg $\chi_3(z) = 1$ and $|deg\chi_9(z) = 3$.
(2) Let $p = 15$ and $n = 7$. Then $a_7(z) = \chi_3(z)\chi_5(z)\chi_{15}(z)$ and
$D_{\xi_3,K(1/15)}(t)$
$= \lambda_{\xi_3,K(1/15)}(t)D_{\xi_3,K(1/3)}(t) = (1+t^2)(1-t^6+t^{12}-t^{18}+t^{24})$,
$D_{\xi_5,K(1/15)}(t) =$
$[\lambda_{\xi_5,K(1/15)}(t)]^2 D_{\xi_5,K(1/5)}(t) = (1-t^{10}+t^{20})^2(1+t^2)(1+t^{10})$.
Since $\prod_{j=3,5,15} D_{\xi_j,K(1/15)}(t) = (1+t^2)(1+t^{30})^6$, it follows
$D_{\xi_{15},K(1/15)}(t)$
$= (1+t^2)(1+t^{30})^6/D_{\xi_3,K(1/15)}(t)D_{\xi_5,K(1/15)}(t)$
$= (1+t^6)(1+t^{10})(1+t^{30})^3/(1+t^2)$.
And hence
$D_{\xi_3,K(1/15)}(\pm 1) = 2, D_{\xi_5,K(1/15)}(\pm 1) = 2^2$ and $D_{\xi_{15},K(1/15)}(\pm 1) = 2^4$.
Note deg $\chi_5(z) = 2$ and deg $\chi_{15}(z)$=4.

Remark 10.5. We learned very recently that D.Silver and S.Williams proved Conjecture 10.1 (1) for 2-bridge knots.

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