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EVERY KNOT IS CLOSE TO INFINITELY MANY HYPERBOLIC KNOTS

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ABSTRACT. We show that every knot in the 3-sphere can be deformed into a hyperbolic knot by a single crossing change without changing knot concordance class and Alexander invariant. Furthermore, each knot has infinitely many such crossing changes producing distinct hyperbolic knots.

1. INTRODUCTION

Following Thurston's uniformization theorem ([14], [8]) and the torus theorem ([3], [4]) every knot K in the 3-sphere S^3 can be classified as:

- a *torus knot*, i.e., a knot which can be placed on a standardly embedded torus,
- a *satellite knot*, i.e., a knot whose exterior contains an incompressible torus which is not boundary parallel, or
- a *hyperbolic knot*, i.e., a knot whose complement admits a complete riemannian metric of constant curvature -1 of finite volume.

Among these hyperbolic knots are most important; and we empirically know that 'most' knots are hyperbolic. It is conjectured in [1] that the proportion of hyperbolic knots among all prime knots with minimal crossing number less than n approaches 1 as $n \to \infty$. In this paper, we demonstrate the abundance of hyperbolic knots by showing that every knot is 'close' to infinitely many hyperbolic knots in terms of crossing change.

We regard that two knots are the same if they are isotopic in S^3. For a knot K in S^3, let $B_n(K)$ be the set of knots each of which is obtained by changing at most n crossings in a diagram of K.

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Theorem 1.1. *For each knot K in S^3, $B_1(K)$ contains infinitely many hyperbolic knots. In particular, an arbitrary knot can be deformed into a hyperbolic knot by a single crossing change.*

In Section 3, we consider torus knots and satellite knots, and raise some questions.

2. Proofs

A (2-string) *tangle* is a pair (B, t) where B is a 3-ball and t is a pair of disjoint arcs properly embedded in B. We call (B, t) a *trivial tangle* if there is a homeomorphism from (B, t) to $(D \times I, \{x, y\} \times I)$, where D is a disk containing x and y in its interior. A tangle (B, t) is said to be *prime* if (i) every 2-sphere in B meeting t transversely in two points bounds a 3-ball in B cutting t in an unknotted spanning arc, and (ii) there is no properly embedded disk in B which separates the two arcs of t. A tangle (B, t) is said to be *simple* if it is prime and $B - t$ contains no incompressible tori. Let K be a knot in S^3. Suppose that S is a 2-sphere meeting K transversely in four points and separating S^3 into two 3-balls B_1 and B_2. Then $(B_i, B_i \cap K)$ are tangles, and we say that K is decomposed into the union of two tangles $(B_i, B_i \cap K)$, where $i = 1, 2$.

Proposition 2.1. *Any knot K in S^3 is decomposed into the union of a simple tangle and a trivial tangle.*

Proof. This follows from Myers [10, Theorem 1.1]. Take an arc c in S^3 such that $c \cap K = \partial c$, and $c' = c \cap E(K)$ is a properly embedded arc in $E(K) = S^3 - \mathrm{int}N(K)$; then $E(K) - \mathrm{int}N(c') \cong S^3 - \mathrm{int}N(K \cup c)$. By [10, Theorem 1.1] we can choose c so that $S^3 - \mathrm{int}N(K \cup c)$ is boundary-irreducible and contains no incompressible tori. Let us choose a small regular neighborhood B_1 of c in S^3 so that $(B_1, B_1 \cap K)$ is a trivial tangle. The 2-sphere ∂B_1 decomposes K into the union of the trivial tangle $(B_1, B_1 \cap K)$ and the tangle $(B_2, B_2 \cap K)$ where $B_2 = S^3 - \mathrm{int}B_1$. Since $S^3 - \mathrm{int}N(K \cup c) = B_2 - \mathrm{int}N(B_2 \cap K)$, $(B_2, B_2 \cap K)$ is a simple tangle. □(Proposition 2.1)

Proof of Theorem 1.1. By Proposition 2.1, K is decomposed into the union of a trivial tangle $(B_1, B_1 \cap K)$ and a simple tangle $(B_2, B_2 \cap K)$. Isotope the trivial tangle $(B_1, B_1 \cap K)$ fixing its boundary as in Figure 1.

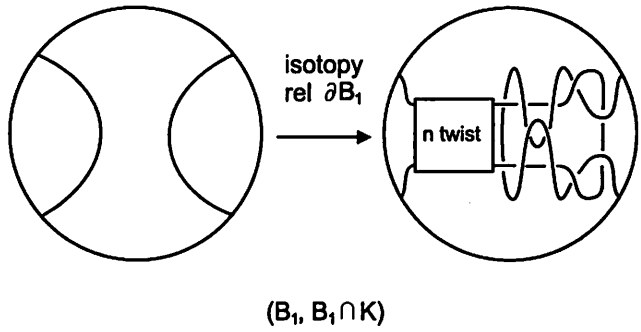

FIGURE 1

Let K_n be the knot obtained from K by changing a crossing of K as described in Figure 2.

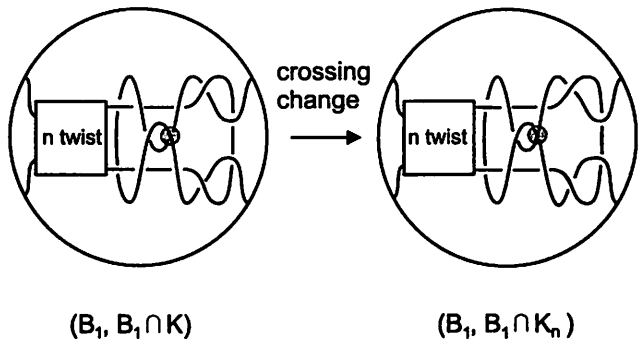

FIGURE 2

The tangle $(B_1, B_1 \cap K)$ is changed to $(B_1, B_1 \cap K_n)$. Soma [13, Lemma 3] proved that $(B_1, B_1 \cap K_n)$ is a simple tangle. It follows that K_n is decomposed into the union of two simple tangles $(B_i, B_i \cap K_n)$, $i = 1, 2$, where $(B_2, B_2 \cap K_n) = (B_2, B_2 \cap K)$. Applying [13, Theorem 1], we see that K_n is a simple knot, i.e., every incompressible torus in the exterior is boundary parallel. Thus K_n is a torus knot or a hyperbolic knot. The first alternative implies that a torus knot is decomposed into the union of two prime tangles, but this contradicts [2, Theorem 2.1]. It follows that K_n is a hyperbolic knot.

It remains to show that $\{K_n\}_{n\in\mathbb{Z}}$ contains infinitely many distinct knots. Note that K_n is obtained from K_0 by twisting n times along the disk D in Figure 3.

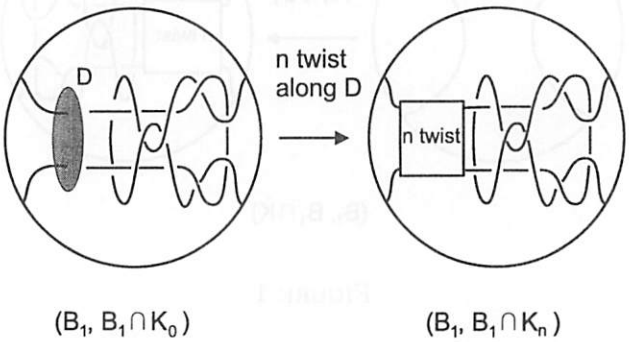

FIGURE 3

Claim 2.2. *The circle ∂D does not bound a disk D' in S^3 which intersects K_0 in at most one point.*

Proof. Since the algebraic intersection number of K_0 and D is zero, it is sufficient to show that there is no disk D' satisfying $\partial D' = \partial D$ and $D' \cap K_0 = \emptyset$. Suppose for a contradiction that we had such a disk D'. Let A be an obvious annulus in $B_1 - B_1 \cap K_0$ connecting ∂D and an essential simple loop in $\partial B_1 - \mathrm{int}N(B_1 \cap K_0)$. Then the existence of a (possibly singular) disk $A \cup D'$ would imply the compressibility of $F = \partial B_1 - \mathrm{int}N(B_1 \cap K_0)$ in $E(K_0)$. This is a contradiction. □(Claim 2.2)

By applying [7, Theorem 3.2] we see that $\{K_n\}_{n\in\mathbb{Z}}$ consists of infinitely many knots. This completes the proof of Theorem 1.1. □(Theorem 1.1)

Remark. In Theorem 1.1 we can choose a crossing change of K so that the resulting hyperbolic knot is concordant to K and has the same Alexander invariant as K. In fact, by [12, Lemma 3.3] the crossing change given in the proof of Theorem 1.1 does not change knot concordance class and Alexander invariant.

3. QUESTIONS

In contrast to hyperbolic knots, torus knots are scattered.

Proposition 3.1. *For any knot K, $B_1(K)$ contains only finitely many torus knots. If K is a knot such that the signature $\sigma(K)$ is 0 and the unknotting number $u(K)$ is 3 (the connected sum of three copies of the figure eight knot is an example), then $B_1(K)$ contains no torus knots.*

Proof. We use the following theorem of Kronheimer and Mrowka [6]: for (p,q)-torus knots $T_{p,q}$, where $|p| > q > 0$, $u(T_{p,q}) = \dfrac{(|p|-1)(q-1)}{2}$. If a torus knot $T_{p,q}$ is contained in $B_1(K)$, then $|u(T_{p,q}) - u(K)| \leq 1$. By the above fact there are only finitely many pairs (p,q) satisfying this inequality. The first assertion is thus proved.

Let K be a knot with $u(K) = 3$ and $\sigma(K) = 0$. If $T_{p,q}$ were contained in $B_1(K)$, then $u(T_{p,q}) = 2, 3$, or 4. Then Kronheimer and Mrowka's theorem implies that $(p,q) = (\pm 4, 3), (\pm 5, 2), (\pm 5, 3), (\pm 7, 2), (\pm 9, 2)$. None of the corresponding torus knots has signature 0 or ± 2; see the table on p. 297 of [5]. Since $\sigma(K) = 0$, this contradicts the fact that a single crossing change to any knot changes its signature by 0 or ± 2 [9].

Let K be the connected sum of three copies of the figure eight knot. The minimal number of the generators of the Alexander invariant of K is three. Thus, $u(K) \geq 3$ by [11]. It follows $u(K) = 3$ by inspection. □(Proposition 3.1)

For each knot K, $B_1(K)$ surely contains a satellite knot. For example, we obtain a satellite knot by pulling a subarc of K around in a knotted manner and then crossing K once. However, the minimal crossing number of the resulting satellite knot seems to be bigger than that of K. In general, a minimal crossing diagram will not admit a single crossing change yielding a satellite knot. We here raise the following questions.

Questions. (1) Let K be a knot with minimal crossing number large enough. Does $B_1(K)$ contain a hyperbolic knot (a satellite knot) whose minimal crossing number does not exceed that of K?

(2) Let K be a prime, satellite knot. Does a single crossing change to a minimal crossing diagram of K yield a hyperbolic knot?

Remark. On minimal crossing diagrams of some torus knots and some composite alternating knots, no single crossing change yields a hyperbolic knot.

(3) For any knot projection G of n crossings, we can obtain 2^n knot diagrams by indicating over-under relation at each crossing point in G. Let $f(G)$ be the proportion of diagrams

representing non-hyperbolic knots among all the knot diagrams obtained from G. For any number r with $0 < r < 1$, $g(r, n)$ denotes the proportion of knot projections of n crossings with $f(G) > r$ among all knot projections of n crossings. Then does $g(r, n)$ tend to 0 as $n \to \infty$?

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