HAKONE SEMINAR 26 (2010) 51-61.

O-GRAPHIC STUDY OF CLOSED 3-MANIFOLDS

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INTRODUCTION

In [1, 2], Benedetti and Petronio defined a finite graphic presentation of an oriented 3-manifold called *o-graphs*. In this brief survey, we review properties of o-graphs of closed oriented 3-manifolds. In particular, we focus on an analogy between o-graphs and knot diagrams. Refer [2] for terminology about branched spines and o-graphs. Note that the construction and decomposition of closed oriented 3-manifolds in terms of o-graphs, explained in this paper, play a key role for the computation of quantum invariants, e.g. colored Turaev-Viro invariants of knots in a closed oriented 3-manifold.

We remind that if P is a branched spine of an oriented M then P also carries an orientation, defined as a screw-orientation along the edges of the singulatiries $S(P)$ of P with a natural compatibility at vertices (see [1]). Conversely, if P is a branched standard polyhedron, then P is orientable, and the manifold it defines is oriented. In addition, P can be described by two additional structures on the 4-valent graph $S(P)$:

(1) An embedding in the plane of the neighbourhood of each vertex, with two opposite strands marked as being over the other two, as in knot projections;
(2) Orientation of each edge such that the orientations of opposite edges match through the vertices.

A 4-valent graph with these additional structures is called a *normal o-graph*. Figure 1.3 illustrates an example of a normal o-graph. Note that normal o-graphs are special kind

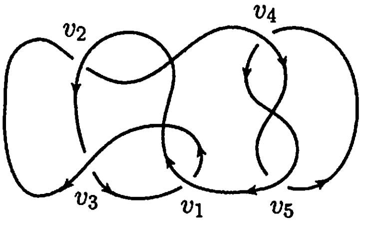


FIGURE 1. A normal o-graph.

of o-graphs. It was shown in [1] that any normal o-graph defines an branched standard polyhedron, whence an oriented manifold, and that two normal o-graphs defining the same branched polyhedron are related by certain moves called *C-moves*. Define the *first* and the *second* vertex of an edge of a normal o-graph using the orientation.

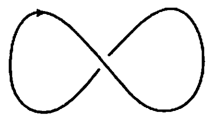


FIGURE 2. A normal o-graph of S^3.

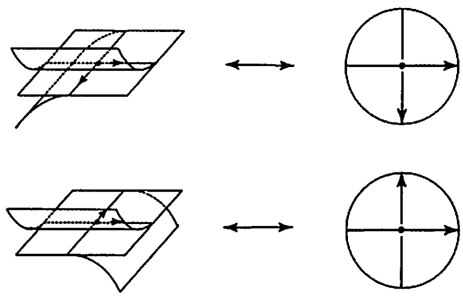


FIGURE 3. Branched spines and normal o-graphs.

1. CONSTRUCTIONS AND DECOMPOSITIONS OF NORMAL O-GRAPHS

In this section, we investigate operations of closed oriented 3-manifolds, e.g. connected sums and torus decompositions, by means of normal o-graphs.

1.1. Normal o-tangles.

Definition. A graph Γ is called an *normal o-tangle* of n components if

(1) The degrees of all but $2n$ vertices $v_1, v_2, \dots, v_{2n}$ of Γ are 4, and the those of the $2n$ distinguished vertices are 1;
(2) Γ is equipped with the following structures:
  (a) A pairing $\{v_{2i-1}, v_{2i}\}$, $i = 1, 2, \dots, n$ of the degree 1 vertices;
  (b) An embedding of Γ in the plane of the neighbourhood of each vertex, with two opposite strands marked as being over the other two, as in knot projections;
  (c) Orientation of each edge such that the orientations of opposite edges match through the vertices and that one of $\{v_{2i-1}, v_{2i}\}$, $i = 1, 2, \dots, n$, is the first vertex of the unique edge connecting it and the other is the second one of the unique edge connecting it.

We defined the normal o-tangles Γ_L, Γ_R, $\Gamma_{\overline{L}}$, $\Gamma_{\overline{R}}$ of 2 components and a normal o-tangle Γ_C of 1 component as shown in Figure 5. For each word $w = X_1 X_2 \cdots X_n$ of letters $L, R, \overline{L}$ or $\overline{R}$, we define a normal o-tangle $\overline{\Gamma}_w$of 1 component, called a *tail* with respect to w, by the following inductive manner:

For a letter X_1 of length one, the normal o-tangle $\overline{\Gamma}_{X_1}$ of 1 component is defined by gluing the end points $1, 2$ of Γ_C to the end points $3, 4$ of Γ_{X_1}, respectively. Assume the piece of an o-graph $\overline{\Gamma}_{X_1 X_2 \cdots X_{i-1}}$ is constructed. Then the piece $\overline{\Gamma}_{X_1 X_2 \cdots X_i}$ is defined by

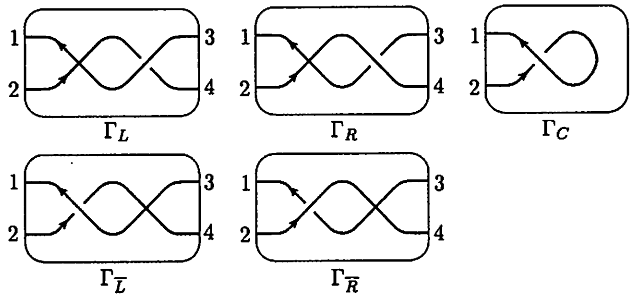


FIGURE 4. The normal o-tangles Γ_L, Γ_R, $\Gamma_{\overline{L}}$, $\Gamma_{\overline{R}}$ and Γ_C.

gluing the end points 1, 2 of $\overline{\Gamma}_{X_1X_2\cdots X_i}$ to the end points 3, 4 of Γ_{X_i}, respectively, see Figure 5.

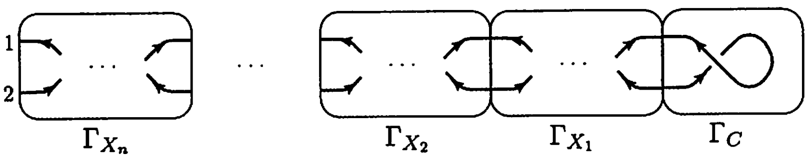


FIGURE 5. The tail $\overline{\Gamma}_{X_1X_2\cdots X_n}$.

For a pair of mutually coprime positive integers (p, q), $p > q > 0$, we define a word $w(p, q)$ of letters $L, R, \overline{L}, \overline{R}$ so that

$$w(p,q) := \begin{cases} L^{a_1}\, R^{a_2}\, L^{a_3} \cdots L^{a_{n-2}} R^{a_{n-1}}\, L^{a_n} & (n\text{: odd}) \\ L^{a_1}\, R^{a_2}\, L^{a_3} \cdots R^{a_{n-2}} L^{a_{n-1}}\, R^{a_n} & (n\text{: even}) \end{cases},$$

where

$$q/p = [a_1, a_2, \ldots, a_n, 1] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + 1}}}}.$$

In the following, we denote the normal o-tangle $\overline{\Gamma}_{w(p,q)}$ of 1 component with respect to the word $w(p, q)$ simply by $\Gamma_{(p,q)}$.

1.2. **Connected sums.** Given diagrams of two knots K_1 and K_2 in the 3-sphere. Then a diagram of the connected sum $K_1\#K_2$ of K_1 and K_2 is obtained only in tems of their diagrams. We have an analogous fact on normal o-graphs. Let Γ_1 and Γ_2 be normal o-graphs of closed oriented 3-manifolds M_1 and M_2. Consider the two normal o-graphs Γ_1 and Γ_2 are on a plane and suppose these graphs are disjoint. Find a rectangle in the plane where one pair of sides are edges of each o-graph but is otherwise disjoint from the o-graphs and so that the edges on the sides of the rectangle are oriented around the boundary of the rectangle in the same direction. Replace the o-tangle of 2 components lying on a neighborhood of the rectangle as shown in Figure 6. Denote by $\gamma_1\#\Gamma_2$ be the resulting normal o-graph.

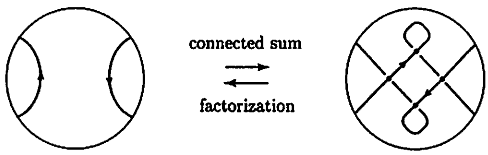


FIGURE 6. Connected sum.

Theorem 1.1 ([8]). *The o-graph $\Gamma_1 \# \Gamma_2$ is of the closed oriented 3-manifold $M_1 \# M_2$.*

1.3. **Torus decompositions and Dehn fillings.** Let M be a connected, compact, orientable 3-manifold such that ∂M is a torus. A *slope* on ∂M is an isotopy class of unoriented, essential, simple closed curves in ∂M. A slope r determines a first homology class $H_1(\partial M)$, well-defined up to sign, obtained by orienting a representative curve for r and considering the homology class of the associated 1-cycle. Conversely, any element of $H_1(\partial M)$ can be represented by a nonseparating, oriented, simple closed curve. This curve is well-defined up to isotopy, and so corresponds to some slope r on ∂M. We use the symbol $\alpha(r)$ to represent either of the two homology classes in $H_1(\partial M)$ associated to a slope r. Fix a slope r on ∂M and let $M(r)$ denote the manifold obtained by attaching a solid torus to M in such a way that the meridional slope on the boundary of the solid torus is identified with r. We say that $M(r)$ is the *Dehn filling M* along ∂M with slope r. It is well-known [13, 10] that any closed orientable 3-manifold results from filling the exterior of some link in the 3-sphere.

Recall that a graph is said to be 2-*connected* if is not connected after removing appropriate two edges. For a graph G, we denote by $V(G)$ the set of vertices of G, and by $E(G)$ the set of edges of G. Also, for a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G, as usual.

Lemma 1.2. *A connected 4-regular graph is 2-connected.*

Proof. Let G be a connected 4-regular graph. Assume that G is not 2-connected. Then there is an edge e such that $G \setminus e$ has two components G_1 and G_2, and we have

$$2 \cdot \#E(G_1) = \sum_{v \in V(G_1)} \deg_{G_1}(v) = 4 \cdot \#V(G_1) - 1.$$

This is impossible. □

By the proof of above lemma, we see that if a 4-regular graph is k-connected for an odd number $k \in \mathbb{N}$, then it is also $(k+1)$-connected.

Definition. We say that an o-graph is *decomposable* if it is not 4-connected. If an o-graph Γ is decomposable, there exists an embedded circle on the plane which intersects Γ transversely twice. We call this circle a *decomposing circle* for Γ.

Note that a decomposing circle C for Γ separates Γ into two normal o-tangls of 1 component.

Lemma 1.3. *Let $(S; \alpha, \beta)$ be a Heegaard diagram of a closed 3-manifold M. If there is a separating simple closed curve $\gamma \subset S$ such that*

(1) *γ is in general position with respect to $\alpha \cup \beta$;*
(2) *$\alpha \cap \gamma = \emptyset$;*
(3) *there are two integers $1 \leqslant i < j \leqslant g$ such that $\#(\beta_i \cap \gamma) = \#(\beta_j \cap \gamma) = 2$ and that $\beta_k \cap \gamma = \emptyset$ for $k \neq i, j$;*
(4) *if we set $\beta_i \cap \gamma = \{p_1, p_2\}$ and $\beta_j \cap \gamma = \{q_1, q_2\}$, they appear, without loss of generality, in the order as p_1, q_1, p_2, q_2 along the loop γ, i.e. γ intersects β_i and β_j alternatively, and*
(5) *at the points p_1 and p_2 (resp. q_1 and q_2), γ and β_i (resp. β_j) intersects with different signs,*

then there is a torus T embedded in M such that $T \cap S = \gamma$.

Proof. Since the simple closed curve γ does not intersect α, γ bounds a 2-disk E_α in the handlebody H_α.

Consider the handlebody H_β. Since γ intersects β_i and β_j twice, respectively, γ separates β_i into two arcs a_1 and a_2, and β_j into b_1 and b_2. Let a^+ and a^- be components of $\partial N(\beta_i; S) \backslash \gamma$ corresponding to the arc a_1. Let b^+ and b^- be components of $\partial N(\beta_j; S) \backslash \gamma$ corresponding to the arc b_1. Due to the conditions (4) and (5), the union

$$\gamma' = (\gamma \setminus (N(\beta_i; S) \cup N(\beta_j; S))) \cup a^+ \cup a^- \cup b^+ \cup b^-$$

is a simple closed curve on the sphere $\partial \overline{(H_\beta \setminus \sum_k D_{\beta_k})}$, and hence it bounds a disk E_β in $\overline{H_\beta \setminus \sum_k D_{\beta_k}}$.

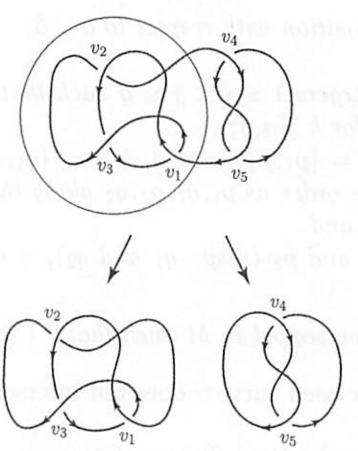
The disk E_β can be regarded as a properly embedded disk in H_β. Now, identifying the boundary arcs a^+ with a^- and b^+ with b^- by naturally retracting the neighborhood $N(\beta_i; S)$ and $N(\beta_j; S)$ to β_i and β_j, respectively, we get a once punctured torus T_β such that $\partial T_\beta = \gamma$ by condition (5). Then the union $E_\alpha \cup T_\beta$ becomes a required embedded torus. □

As a corollary of the above Lemma 1.3 and Theorem 5.14 in [8], we have the following.

Theorem 1.4. *Let Γ be a normal o-graph of a closed 3-manifold. Assume that Γ is decomposable. Let C be a decomposing circle. Let Γ_1 and Γ_2 be normal o-tangle obtained by splitting Γ along C. Then there exists an embedded torus T satisfying the following properties:*

(1) *T separates M into two compact 3-manifolds M_1 and M_2;*
(2) *For each $i \in \{1, 2\}$, there exists a basis $\{\gamma_1, \gamma_2\}$ of $H_1(\partial M_i)$ such that filling Γ_i by a tail $\Gamma_{(p,q)}$ gives a normal o-graph of $M_i(r)$, where r is a slope on ∂M_i with $\alpha(r) = p\gamma_1 + q\gamma_2$; and*
(3) *The torus T is a union of orbits of the flow carried by a flow-spine which has the o-graph Γ.*

Remark that the first assertion of the above theorem is announced by Kouno in [9] in the category of almost-special spines. The above observation insists that we can compare a piece of o-graphs with two end points to a tangle of knots. Actually, the operation in Theorem 1.6 (3) corresponds to *tangle surgery* for knots, see e.g. [7].



1.4. **Dehn surgery on knots in the 3-sphere.** Let P be a flow-spine of the 3-sphere. For each edge e of $S(P)$, there is 3-germs of regions of P. Two of them are on the same side of the edge, and let R be the upper region.

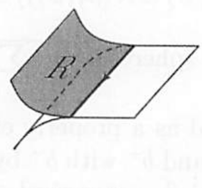


FIGURE 7. The region R.

Let $e(K)$ be the knot such that $K \cap P = K \cap \operatorname{Int} R$ and $K \cap P$ is a single transverse intersection.

Theorem 1.5. *For an arbitrary knot $K \in S^3$, there is a flow-spine and its edge e such that K is ambient isotopic to $e(K)$.*

Theorem 1.6. *Let Γ be a normal o-graph of the 3-sphere. Let e be an edge of Γ. Let Γ_e be a normal o-tangle of 1 component obtained by cutting Γ at an interior point of e. Then there exists a basis $\{\gamma_1, \gamma_2\}$ of $H_1(\partial N(K))$ such that filling Γ_e by a tail $\Gamma_{(p,q)}$ gives a normal o-graph of $K(r)$, where r is a slope on $\partial N(K)$ with $\alpha(r) = p\gamma_1 + q\gamma_2$.*

The above theorems implies that any Dehn surgery along a knot can be interpreted as 'a 'simple" tangle surgery of an o-graph.

1.5. **Seifert fibered 3-manifolds.** Let g be a non-negative integer and b be an integer. Let $(p_1, q_1), (p_2, q_2), \dots, (p_r, q_r)$ be pairs of mutually coprime integers such that $1 < p_i$ and $0 < q_i < p_i$ $(i = 1, 2, \dots, r)$.

Assume that $g + r > 2$. Prepare $g + r - 2$ copies $\Gamma_S^1, \Gamma_S^2, \dots, \Gamma_S^{g+r-2}$ of the o-tangle Γ_S and g copies $\Gamma_T^1, \Gamma_T^2, \dots, \Gamma_T^g$ of the o-tangle Γ_T, where the o-tangles Γ_S and Γ_T are depicted in 8. First, attach the o-tangle Γ_b to the boundaries of the o-graph Γ_S^1 so that

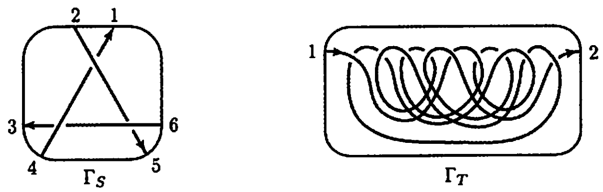


FIGURE 8. The normal o-tangles Γ_S and Γ_T.

the degree one vetrices 1 and 2 of $\partial\Gamma_S^1$ match the vertices 1 and 2 of $\partial\Gamma_b$. For odd k with $1 \leqslant k \leqslant r$, attach the o-tangle $\Gamma_{(p_k,q_k)}$ to the o-tangle Γ_S^k so that the degree one vertices 3 and 4 of $\partial\Gamma_S^1$ match the vertices 1 and 2 of $\partial\Gamma_{(p_k,q_k)}$. For even k with $1 \leqslant k \leqslant r$, attach the o-tangle $\Gamma_{(\alpha_k,p_k-q_k)}$ to the degree one vertices of the o-tangle Γ_S^k in the same manner as above. For $1 \leqslant k \leqslant g-2$, attach the o-tangle Γ_T^k to the degree one vertices of the piece Γ_S^{r+k} in the same manner as above. Attach Γ_T^{g-1} and Γ_T^g to the degree one vertices 3,4 and 5,6 of the o-tangle Γ_S^k, respectively, in the same manner as above. Note that now we have $g+r-2$ components of o-tangles $\Gamma_1, \Gamma_2, \ldots, \Gamma_{g+r-2}$ such that Γ_k contains Γ_S^k. For even k with $1 \leqslant k \leqslant g+r-2$, change the fixed direction of the edges of the o-tangle Γ_k. Now we get an o-graph by attaching the degree one vertices 3,4 of the o-tangle Γ_k to the vertices 1,2 of the o-graph Γ_{k+1} for $1 \leqslant k \leqslant g+r-2$. We denote it by $\Gamma_{(g;b;(p_1,q_1),(p_2,q_2),\ldots,(p_r,q_r))}$.

If $g+r \leqslant 2$, attach the o-tangle $\overline{\Gamma}_b$ to the degree one vertices of the o-tangle $\overline{\Gamma}_S^1$. Moreover, attach the rest of the o-tangles $\overline{\Gamma}_{(p_i,q_i)}$ and copies of $\overline{\Gamma}_T$. In particular, if $g+r<2$, attach the copies of Γ_C to all the left degree one vetices of $\overline{\Gamma}_S^1$.

Theorem 1.7 ([12]). *The o-graph $\Gamma_{(g;b;(p_1,q_1),(p_2,q_2),\ldots,(p_r,q_r))}$ is of a closed oriented Seifert fibered 3-manifold with Seifert parameter $S(g;b;(p_1,q_1),(p_2,q_2),\ldots,(p_r,q_r))$.*

The construction of an o-graph of a Seifert fibered 3-manifold in Theorem 1.7 gives a good example of Theorem 1.6 (i). Recall that the pieces of an o-graph in the construction correspond to either (trice punctured sphere) $\times S^1$, (once punctured torus) $\times S^1$ or a fibered solid torus. When we connect two pieces, the connection is made by two edges which separate the resulting o-graph and this process corresponds to a gluing along boundary tori. This is a special case of Theorem 1.6.

Fig. 9 illustrates the same o-graph as Fig. and embedded tori $T_1, T_2, T_3, T_4, T_5, T_6$. The tori T_1, T_2, T_3, T_5 bound a solid torus in the manifold $S(1;0;(2,1),(3,1))$, in fact, they consist of fibered in fibered solid tori. The torus T_4 also bounds a solid torus of type $(2,1)$. The torus T_6 separates the manifold $S(1;0;(2,1),(3,2))$ into a Seifert fibered manifold with base surface S^2 and 2 singular fibers and $T^2 \times S^1$.

2. INVARIANTS VIA NORMAL O-GRAPHS

In this section, we formally define two well-known invariants, Heegaard genera and fundamental groups, of closed oriented 3-manifolds in terms of o-graphs.

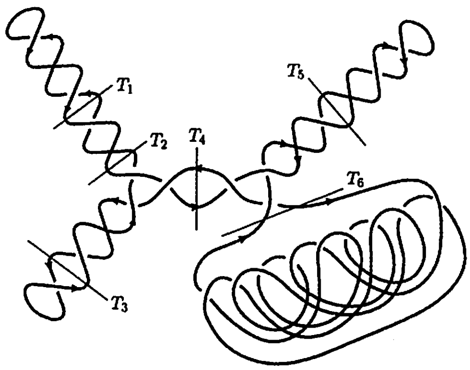


FIGURE 9. Embedded tori in $\Gamma_{(1;0;(2,1),(3,2))}$.

2.1. **Heegaard genus.** A diagram $\mathcal{D}$ of a knot $K \subset S^3$ is a projection of K into a plane such that at most two strands of the knots intersect at any point and that there are finitely many such intersections. By convention, at each crossing of two strands, one removes a segment of the projected image of the lower strands to convey relative height information. These breaks make the diagram a set of disjoint arcs.

Definition. An arc c is a piece of a knot projection or an o-graph which begins at one under-crossing and ends at the next one moving along the the knot diagram or the o-graph. An arc c is called an *overbridge* if it contains at least one over-crossing in between the two endpoints, which is under-crossings.

As the *bridge number* of a diagram of a knot K is defined to be the number of overbridges included in the diagram, the *block number* of an o-graph of a closed orientable 3-manifold M is defined to be the number of overbridges included in the o-graph. Then the *block number* $\mathrm{BL}(M)$ of a closed oriented 3-manifold M is defined to be the minimum of the block numbers of all o-graphs of M, while the *bridge number* of a knot K is defined to be the minimum of the bridge numbers of all diagrams of K. It is proved that $\mathrm{Bl}(S^3) = 1$ and $\mathrm{Bl}(S^2 \times S^1) = 0$.

Theorem 2.1 ([4, 8]). *Every closed orientable 3-manifold M except $S^2 \times S^1$ and S^3 satisfies* $\mathrm{Bl}(M) = \mathrm{HG}(M)$.

2.2. **Fundamental groups.** Joyce [6] and Matveev [11] independently introduced an invariant of knots called the *knot quandle* and the *distributive groupoid*, respectively. The knot quandleis defined by associating a quandle structure to a knot. A quandle is a set Q equipped with a binary operation $\circ : Q \times Q \to Q$ satisfying the following three axioms:

(1) For all $x \in Q$, $x \circ x = x$;
(2) For all $y, z \in Q$, there exists a unique element $x \in \Lambda$ such that $x \circ y = z$;
(3) For all $x, y, z \in Q$, $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$.

Let $\mathcal{D}$ be a diagram of an oriented knot K. Note that an orientation of the knot naturally define the orientations of these arcs. Recall that two diagrams represent the same knot if and only if they are connected by a finite sequence of Reidemeister moves. Let Q_K be the set of arcks of $\mathcal{D}$. For each crossing, define a retion $x \circ y = z$ as shown in Fig. 10.

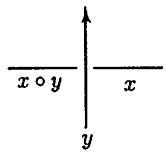


FIGURE 10. Knot quandle.

It is shown that Q_K is in fact invariant under Reidemeister moves, see Figure 11. This

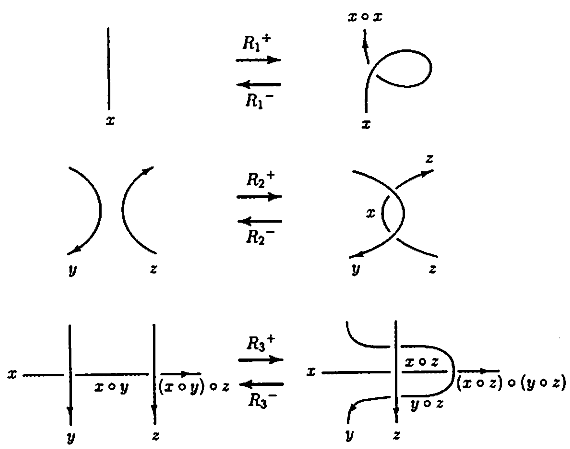


FIGURE 11

implies that the Q_K is a knot invariant.

Given an o-graph of a closed 3-manifold, we can obtain its fundamental group following the argument of the knot quandle. Let Γ be an o-graph of a closed 3-manifold M, G be the free groups generated by the elements corresponds to the labels on the arcs of Γ and R be relations of elements of G given as shown in Fig. 12.

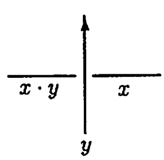


FIGURE 12

Proposition 2.2. *Under the above setting, we have $\pi_1(M, *) = \langle G \mid R \rangle$.*

Note that we have an analogous argument on the invariance of this group under fundamental moves for o-graphs.

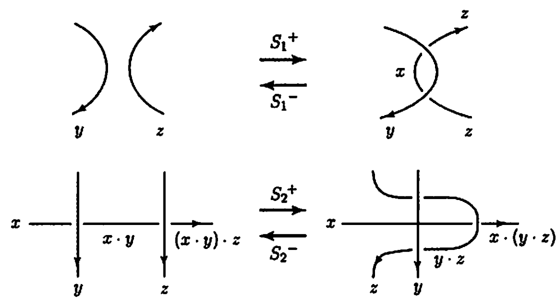


FIGURE 13

Example. Fig. 14 shows an o-graph of the *quaternion space*, which is 3-fold cyclic branched covering of S^3 branched over a trefoil knot.

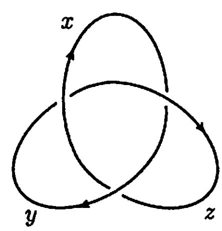


FIGURE 14

Then we can easily obtain the presentation $\langle x, y, z \mid xy = z, yz = x, zx = y \rangle$ of its fundamental group following the rule shown in Fig. 12.

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