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COLORED TURAEV-VIRO INVARIANTS OF TWIST KNOTS

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ABSTRACT. In the present paper we give a formula for colored Turaev-Viro invariants of twist knots using special polyhedra derived from $(1,1)$-decomposition of the knots.

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INTRODUCTION

In the recent paper [1], Barrett, Garcia-Islas and Martins defined a new series of invariants, *colored Turaev-Viro invariants*, of a pair (M, L), where M is a closed oriented 3-manifolds and L is an oriented link embedded in M. Pervova and Petronio [14] described these invariants using a good position (see Section 1) of a link L with respect to a special polyhedron P in a 3-manifold M. The invariants are then defined as state-sums on such special polyhedron, restricting only to states such that the regions intersecting L have a certain pre-fixed color. The paper [14] provided links in S^3 which have the same HOMFLY polynomial and and the same Kauffman polynomial but distinct colored Turaev-Viro invariants. This fact implies that colored Turaev-Viro invariants have somewhat different characteristic from skein-invariants of links.

In this paper, we construct special spines (*o-spines*) for twist knots using $(1,1)$-decomposition of them, and then we give a formula for colored Turaev-Viro invariants of twist knots using these spines.

1. SPINES AND LINKS

The following terminology is due to [11, 15]. A *simple polyhedron* P is defined to be a compact polyhedron such that the link of each point of P can be embedded into the complete graph Γ_4 with 4 vertices. A *vertex* of simple polyhedron is a point with the link homeomorphic to Γ_4. We denote by $V(P)$ the set of all vertices of P.

A simple polyhedron P is said to be *almost-special* if every point in P has a neighborhood homeomorphic to one of the three models shown in Fig. 1. The points of type (ii) and (iii) are said to be *singular* and the set of singular points is denoted by $S(P)$. The closure of each component of $P \setminus V(P)$ is called an *edge*. An almost-special polyhedron is said to be *special* if there is no loop in $S(P)$ and $P \setminus S(P)$ consists of disks. The closure of each component of $P \setminus S(P)$ is called a *region* and we denote by $R(P)$ the set of all regions of P.

A *spine* P of a closed orientable 3-manifold M is a simple polyhedron in M such that $M \setminus P$ is an open 3-ball. See [2, 11] for details of spines.

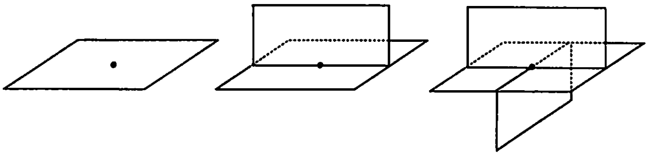

FIGURE 1. Neighborhoods of points in a simple polyhedron.

Let L be a link in a closed 3-manifold M. Let P be an almost-special polyhedron in M. Then L is said to be *in general position with respect to* P if L intersects P transversely at points in $P \setminus S(P)$. See [14] for orientations of links, special polyhedrons and 3-manifolds, and their compatibility. Let L be an oriented link in a closed oriented 3-manifold M. Then a marked oriented special polyhedron P of M, i.e. an oriented special polyhedron with some marked regions, is called an *o-spine* if it satisfies the following conditions:

(1) P and M are consistently oriented;
(2) in each connected component B of $M \setminus P$, the pair $(B, B \cap L)$ is homeomorphic (as a pair) to either $(\mathrm{Int}B^3, \emptyset)$ or $(\mathrm{Int}B^3, I)$, where I is an unknotted arc in $\mathrm{Int}B^3$;
(3) each component of L intersects P once and transversely; and
(4) the marked regions of P are precisely those which intersect L, and they are oriented consistently with the orientation of L and M,

The region α of an o-spine of L which intersects a component of L is said to be *dual* to the component.

2. Colored Turaev-Viro invariants of links

2.1. Definitions. Let $\mathbb{K}$ be a commutative ring with unity. An initial datum for a Turaev-Viro invariant consists of

- a finite set I;
- a function $I \to \mathbb{K}^\times$ which assigns to each $i \in I$ its *wight* $w_i \in \mathbb{K}^\times$;
- a certain distinguished element $w \in \mathbb{K}^\times$;
- a distinguished set $\mathcal{A}$ of unordered triples of elements of I, called *admissible*; and
- a function $\mathcal{B} \to \mathbb{K}$ which assigns to each 6-tuple (i, j, k, l, m, n) its *6j-symbol*

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix},$$

where

$$\mathcal{B} := \{(i, j, k, l, m, n) \in I^6 \mid (i, j, k), (i, m, n), (j, l, n), (k, l, m) \in \mathcal{A}\}$$

We require that the 6j-symbols satisfy the following symmetries:

(1)

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ l & n & m \end{vmatrix}$$
$$= \begin{vmatrix} i & m & n \\ l & j & k \end{vmatrix} = \begin{vmatrix} l & m & k \\ i & j & n \end{vmatrix} = \begin{vmatrix} l & j & n \\ i & m & k \end{vmatrix}.$$

We also require that the $6j$-symbols, w_i's and w satisfy the *Biedenharn-Elliott identity*, *orthogonality relation*, *strong irreducibility*, etc., as was required in [14]. See also [16]. Such an initial datum is said to be *good.*

We now fix a good initial datum. Let M be a closed 3-manifold and L be a link in M. Denote the components of L by $L_1, L_2, \ldots, L_n$. Let $\xi : \{L_i\}_i \to I$ be a *coloring* of the link components by elements of I. Let P be an o-spine of the pair (M, L), let α_i be the region of P dual to L_i, and let μ be the number of complementary balls having empty intersection with L.

A *coloring* of a special polyhedron P is a map $\eta : R(P) \to I$. A coloring is said to be *admissible* if for any edge e of P the colors of the three germs of regions incident to e form an admissible triple. Denote by $\mathrm{Adm}_\xi(P)$, as in [14], the set of all admissible colorings η of P such that $\eta(\alpha_i) = \xi(L_i)$ for all $1 \leqslant i \leqslant n$.

For $\eta \in \mathrm{Adm}_\xi(P)$ and $v \in V(P)$, set

$$s_\eta(v) := \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix},$$

where i, j, k, l, m, n are the colors of the germs of regions of P incident to v as shown in Fig. 2.

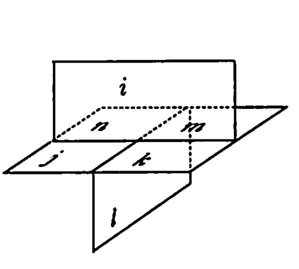

FIGURE 2. A coloring around a vertex.

The colored Turaev-Viro invariant of (M, L, ξ) with respect to the above initial datum is then defined by

$$\mathcal{I}_\xi(P) = w^{-\mu} \sum_{\eta \in \mathrm{Adm}_\xi(P)} \omega(\eta),$$

where the weight $\omega(\eta) \in \mathbb{K}$ of η is defined by

$$\omega(\eta) = \prod_{v \in V(P)} s_\eta(v) \prod_{c \in R(P)} w_{\eta(c)}.$$

We now introduce the initial datum $\mathcal{D}_\varepsilon$ for the colored Turaev-Viro invariants $\mathcal{I}^\varepsilon_*(L)$ of links L defined in [14]. Indeed, it is introduced in [13] to define the *ε-invariant* of 3-manifolds.

Fix a root ε of the equation $x^2 = x + 1$, and an arbitrary square root $\varepsilon^{1/2}$ of ε. The corresponding initial datum $\mathcal{D}_\varepsilon$ consists of the following:

- $\mathbb{K} = \mathbb{C}$;
- $I = \{0, 1\}$;
- the weights are $w_0 = 1$, $w_1 = \varepsilon$, and the the distinguished constant is $w = \varepsilon + 2$;
- $(i, j, k) \in \mathcal{A}$ if and only if

$$i \leqslant j + k,\ j \leqslant k + i,\ k \leqslant i + j;$$

- the value of the $6j$-symbols are given by:

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}_\varepsilon = 1, \begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}_\varepsilon = \varepsilon^{-1/2}, \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}_\varepsilon = \varepsilon^{-1},$$
$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}_\varepsilon = \varepsilon^{-1}, \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}_\varepsilon = -\varepsilon^{-2}.$$

For each color $i \in I$, we denote by $\mathcal{I}_i(M, L)$ the invariant obtained by coloring all the components of L with i. When $M = S^3$, we simply denote the invariant by $\mathcal{I}_i(L)$.

Remark. (a) Let L be a link in a closed 3-manifold M. In [14], it is proved that

$$\mathcal{I}^\varepsilon_0(M, L) = t(E(K)),$$

where $t(M)$ denotes the ε-invariant of M arising from the initial datum $\mathcal{D}_\varepsilon$ and $E(K)$ denotes the complement of L. On the other hand, the paper [14] also proved that there exists a pair of links in S^3 with homeomorphic complements and distinct $\mathcal{I}^\varepsilon_1$ invariants.

(b) The program *Three-manifold Recognizer* [12] calculates Turaev-Viro invariants of link complements in S^3, hence the $\mathcal{I}^\varepsilon_0$ invariants of the link.

2.2. **O-spines of twist knots.** Let $K_{1,1;m}$ $(m \geqslant 1)$ be a knot in S^3 illustrated in Fig. 3. These knots are called *twist knots*. Note that $K_{1,1;1}$ is the figure-eight knot.

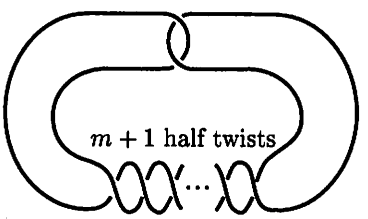

FIGURE 3. The twist knot $K_{1,1;m}$.

Remark. In [10], we defined generalized twist knots $K_{p,q;m}$ in the lens spaces $L(p, q)$. The above notation follows from this viewpoint.

The following is due to the classification of 2-bridge knots and the geometry of their complements, see [5, 3].

Proposition 2.1. *For any $m \geqslant 1$, the knot $K_{1,1;m}$ is hyperbolic.*

We construct an o-spine $P_{1,1;m}$ of the twist knots $K_{1,1;m}$ ($m \in \mathbb{N}$) in the following way. Consider the Heegaard diagram of S^3 shown in the left-hand side of Fig. 4. Here,

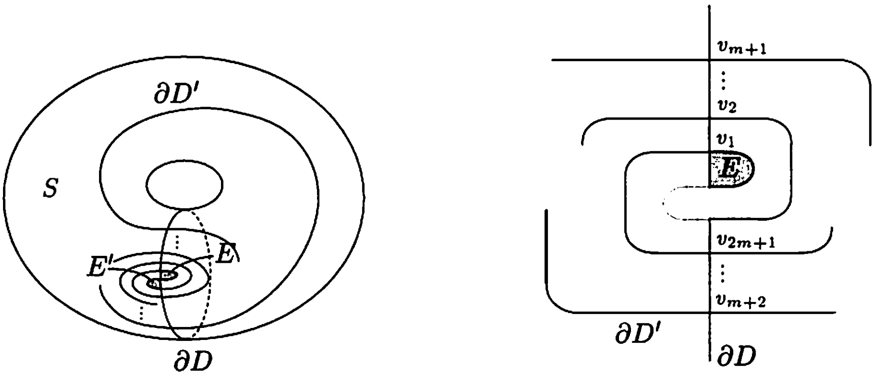

FIGURE 4. Twist knots and Heegaard diagram of S^3.

we assume that $\#\partial D \cap \partial D' = 2m+3$. Set $P_{1,1;m} := (S \cup D \cup D') \setminus E'$, where E' is a component of $S \setminus (D \cup D')$ described in the figure, and denote the vertices of $P_{1,1;m}$ by $v_1, \dots, v_{2m-1}$ as shown in the right-hand side of Fig. 4.

Lemma 2.2 ([10]). *The knot which intersects $P_{1,1;m}$ transversely once at E and which is trivial in $S^3 \setminus P$ is the twist knot $K_{1,1;m}$.*

Proof. We prove the lemma using an example shown in Fig. 5. Consider the knot K

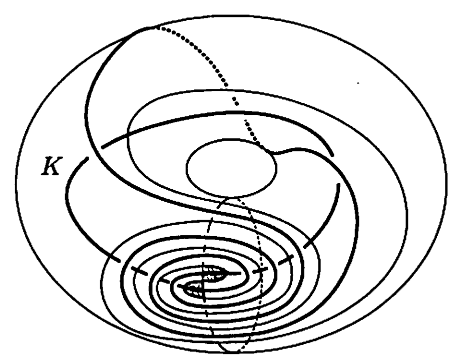

FIGURE 5. The twist knot $K_{1,1;3}$.

such that

(a) K is in $(1,1)$-position with respect to the associated Heegaard splitting;

(b) K intersects E (resp. E') once and transversely; and
(c) K does not intersects $D \cup D'$.

Note that such a knot is uniquely determined (see the bold loop in Fig. 5). Now, $P_{1,1;3} := (S \cup D \cup D') \setminus E'$ is an o-spine of K and Fig. 6 shows that $K \approx K_{1,1;3}$. □

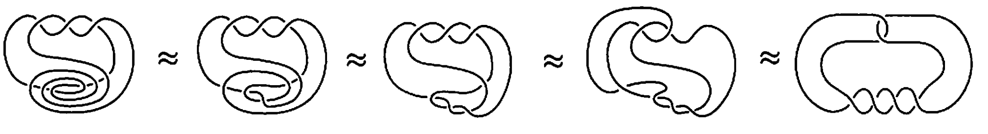

FIGURE 6. The knot K is the twist knot $K_{1,1;3}$.

Lemma 2.2 implies that the special polyhedron $P_{1,1;m}$ is an o-spine of the twist knot $K_{1,1;m}$.

Remark. Since the orientation of knots does not affect the colored Turaev-Viro invariants of them by definition, we do not mention the orientations.

Remark. By the arguments in [8, 9], the above special polyhedron $P_{1,1;m}$ turns out to be a flow-spine [6].

We set

$$R(P_{1,1;m}) = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \ldots, \beta_{m-1}, \gamma_1, \gamma_2, \ldots, \gamma_{m-1}\}$$

as shown in Fig. 7. Note that $E = \alpha_0$, $D \subset \alpha_1$, $D' \subset \alpha_2$ and the other regions are subsets of the Heegaard surface S. Note also that the region α_0 of $P_{1,1;m}$ is dual to the twist knot $K_{1,1;m}$.

2.3. **Examples.** We compute the colored Turaev-Viro invariant $\mathcal{I}_0^\varepsilon$ for the knots $K_{1,1;1}$ and $K_{1,1;2}$ using the o-spines $P_{1,1;1}$ and $P_{1,1;2}$, respectively.

The faces around the vertices of the o-spines $P_{1,1;1}$ and $P_{1,1;2}$ are illustrated in Figs. 8 and 9, respectively.

Lemma 2.3. *Any admissible coloring* $\eta \in \mathrm{Adm}_0(P_{1,1;m})$ $(m = 1, 2)$ *satisfies* $(\eta(\alpha_1), \eta(\alpha_2)) = (0, 0)$ *or* $(1, 1)$.

Proof. Consider the coloring around the vertex v_1. Then the statement follows from the fact that the three faces α_0, α_1 and α_2 share the same edge and the coloring of the face α_0 is initially fixed to be 0. □

Lemma 2.4. *For any admissible coloring* $\eta \in \mathrm{Adm}_0(P_{1,1;m})$ $(m = 1, 2)$ *such that* $(\eta(\alpha_1), (\eta(\alpha_2)) = (1, 1)$, *we have that* $\eta(\alpha_3) = 1$.

Proof. Consider the coloring around the vertex v_2 (resp. v_3) of the o-spine $P_{1,1;1}$ ($P_{1,1;2}$, resp.). Then we see that there is an edge for which the germs of faces consists of α_3, α_3 (appearing twice) and α_1. Since we assume that $\eta(\alpha_1) = 1$, the coloring $\eta(\alpha_3)$ must be 1 in order that the triple $(\eta(\alpha_1), \eta(\alpha_3), \eta(\alpha_3))$ may be admissible. □

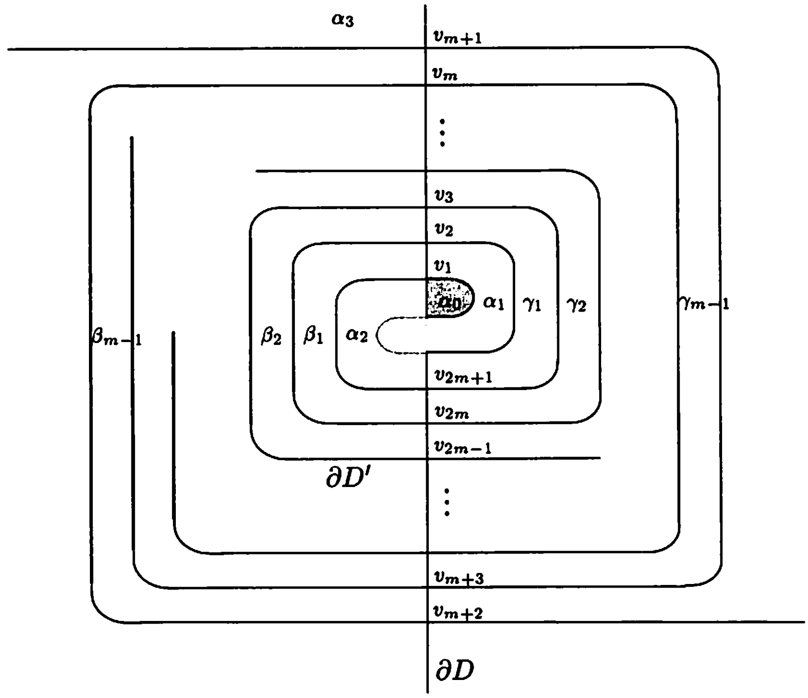

FIGURE 7. The regions of $P_{1,1;m}$.

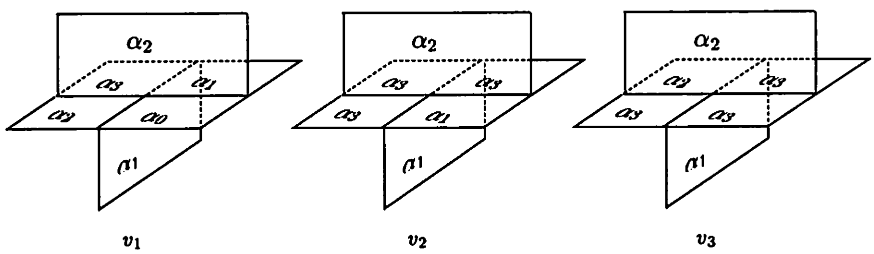

FIGURE 8. The vertices of $P_{1,1;1}$.

Now, we will describe the set $\mathrm{Adm}_0(P_{1,1;1})$ and $\mathrm{Adm}_0(P_{1,1;2})$. By Lemma 2.3 and 2.4, we directly have that $\mathrm{Adm}_0(P_{1,1;1})$ consists of the two elements η_0, η_1, where

$$\begin{aligned}(\eta_0(\alpha_1), \eta_0(\alpha_2), \eta_0(\alpha_3)) &= (0,0,0),\\ (\eta_1(\alpha_1), \eta_1(\alpha_2), \eta_1(\alpha_3)) &= (1,1,1).\end{aligned}$$

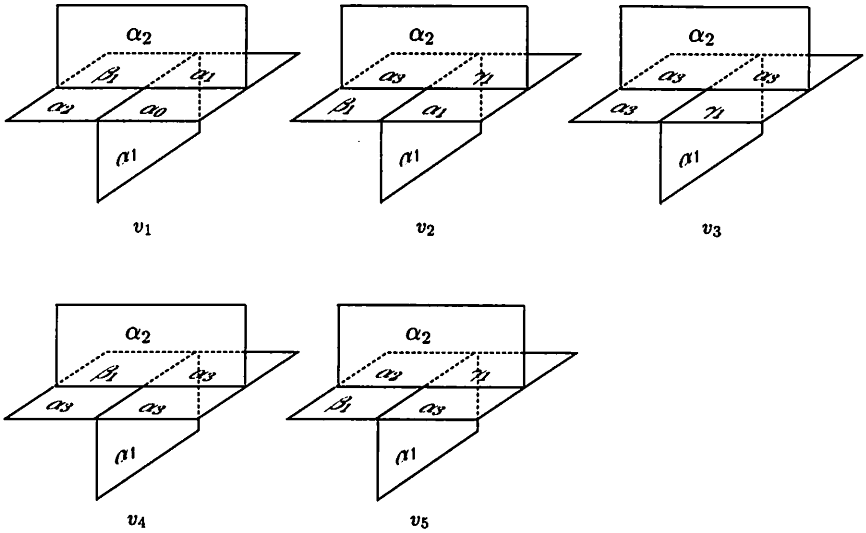

FIGURE 9. The vertices of $P_{1,1;2}$.

Then we get the weights of the vertices as follows:

$$\begin{aligned} s_{\eta_0}(v_1) &= s_{\eta_0}(v_2) = s_{\eta_0}(v_3) = \left| \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right|_{\varepsilon} = 1, \\ s_{\eta_1}(v_1) &= \left| \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \right|_{\varepsilon} = \varepsilon^{-1}, \\ s_{\eta_1}(v_2) &= s_{\eta_1}(v_3) = \left| \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \right|_{\varepsilon} = -\varepsilon^{-2}, \end{aligned}$$

and hence we have

$$\begin{aligned} \mathcal{I}_0^{\varepsilon}(K_{1,1;1}) &= \omega(\eta_0) + \omega(\eta_1) \\ &= s_{\eta_0}(v_1) \cdot s_{\eta_0}(v_2) \cdot s_{\eta_0}(v_3) \cdot \prod_{c \in R(P_{1,1;1})} w_{\eta_0(c)} \\ &\quad + s_{\eta_1}(v_1) \cdot s_{\eta_1}(v_2) \cdot s_{\eta_1}(v_3) \cdot \prod_{c \in R(P_{1,1;1})} w_{\eta_1(c)} \\ &= 1 + \varepsilon^{-1}(-\varepsilon^{-2})^2 \varepsilon^3 \\ &= 1 + \varepsilon^{-2} \\ &= -\varepsilon + 3. \end{aligned}$$

For the o-spine $P_{1,1;2}$, one can check that the set $\mathrm{Adm}_0(P_{1,1;2})$ consists of the five elements η_i $(i = 0,1,2,3,4)$, where

$$\begin{aligned}
(\eta_0(\alpha_1), \eta_0(\alpha_2), \eta_0(\alpha_3), \eta_0(\beta_1), \eta_0(\gamma_1)) &= (0,0,0,0,0),\\
(\eta_1(\alpha_1), \eta_1(\alpha_2), \eta_1(\alpha_3), \eta_1(\beta_1), \eta_1(\gamma_1)) &= (1,1,1,0,0),\\
(\eta_2(\alpha_1), \eta_2(\alpha_2), \eta_2(\alpha_3), \eta_2(\beta_1), \eta_2(\gamma_1)) &= (1,1,1,1,0),\\
(\eta_3(\alpha_1), \eta_3(\alpha_2), \eta_3(\alpha_3), \eta_3(\beta_1), \eta_3(\gamma_1)) &= (1,1,1,0,1),\\
(\eta_4(\alpha_1), \eta_4(\alpha_2), \eta_4(\alpha_3), \eta_4(\beta_1), \eta_4(\gamma_1)) &= (1,1,1,1,1).
\end{aligned}$$

Then we have by elementary calculation that

$$\begin{aligned}
\omega(\eta_0) &= 1,\ \omega(\eta_1) = \varepsilon^{-2},\ \omega(\eta_2) = -\varepsilon^{-2},\\
\omega(\eta_3) &= -\varepsilon^{-2},\ \omega(\eta_4) = \varepsilon^{-4},
\end{aligned}$$

and hence

$$\begin{aligned}
\mathcal{I}_0^{\varepsilon}(K_{1,1;2}) &= 1 - \varepsilon^{-2} + \varepsilon^{-4}\\
&= -2\varepsilon + 4.
\end{aligned}$$

Theorem 2.5. *We have*

$$\mathcal{I}_0^{\varepsilon}(K_{1,1;m}) = \begin{cases} -\varepsilon + 3 & \textit{for } m = 1,6 \pmod{10},\\ -2\varepsilon + 4 & \textit{for } m = 2,5 \pmod{10},\\ -2\varepsilon + 5 & \textit{for } m = 3,4 \pmod{10},\\ 2 & \textit{for } m = 0,7 \pmod{10},\\ 1 & \textit{for } m = 8,9 \pmod{10}, \end{cases}$$

and

$$\mathcal{I}_1^{\varepsilon}(K_{1,1;m}) = \begin{cases} \varepsilon - 2 & \textit{for } m = 1,6 \pmod{10},\\ 2\varepsilon - 3 & \textit{for } m = 2,5 \pmod{10},\\ 2\varepsilon - 4 & \textit{for } m = 3,4 \pmod{10},\\ -1 & \textit{for } m = 0,7 \pmod{10},\\ 0 & \textit{for } m = 8,9 \pmod{10}. \end{cases}$$

Remark. It is worth noting that the above theorem is still valid after we denote the unknot by $K_{1,1;-1}$ and the trefoil knot by $K_{1,1;0}$, see [14].

The following is the first step to prove the above theorem.

Lemma 2.6. *Set*

$$\begin{cases} p_1(\varepsilon) = -\varepsilon + 2,\ p_2(\varepsilon) = -2\varepsilon + 3,\\ q_1(\varepsilon) = -2\varepsilon + 3,\ q_2(\varepsilon) = -3\varepsilon + 5,\\ r_1(\varepsilon) = 5\varepsilon - 8,\ r_2(\varepsilon) = 9\varepsilon - 14, \end{cases}$$

and consider the following recurrence relations:

$$(1)\quad \begin{cases} p_m(\varepsilon) = (-\varepsilon + 2)p_{m-2}(\varepsilon) + 2(\varepsilon - 1)q_{m-2}(\varepsilon) + \varepsilon r_{m-1}(\varepsilon)\ (m \geqslant 3),\\ q_m(\varepsilon) = (-\varepsilon + 2)p_{m-2}(\varepsilon) + (2\varepsilon - 3)q_{m-2}(\varepsilon) - r_{m-1}(\varepsilon)\ (m \geqslant 3),\\ r_m(\varepsilon) = (2\varepsilon - 3)p_{m-2}(\varepsilon) - 2(2\varepsilon - 3)q_{m-2}(\varepsilon) + (-\varepsilon + 2)r_{m-1}(\varepsilon)\ (m \geqslant 3). \end{cases}$$

Then we have

$$\mathcal{I}_0^\varepsilon(K_{1,1,m}) = p_m(\varepsilon)+1,$$
$$\mathcal{I}_1^\varepsilon(K_{1,1,m}) = \varepsilon q_m(\varepsilon),$$

for any $m \in \mathbb{N}$.

From now on, we use the following presentation of the neighborhood of the vertices of the o-spine $P_{1,1;m}$ as shown in Fig. 10.

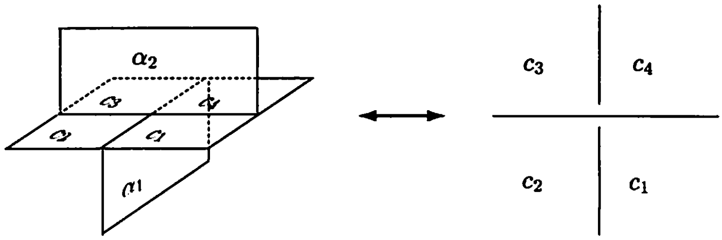

FIGURE 10. Presentation of the neighborhood of a vertex.

The vertices of the o-spine $P_{1,1;m}$ have the neighborhoods as shown in Fig. 11:
Set

$$\mathrm{Adm}_i(P_{1,1;m}) = \{\eta \in \mathrm{Adm}(P_{1,1;m}) \mid \eta(\alpha_0)=i\},$$

for $i = 0, 1$.

We denote by $G_{1,1;m}$ the simple graph whose vertices are the set $R(P_{1,1;m})\backslash\alpha_0,\alpha_1,\alpha_2,\alpha_3$, and where two distinct vertices are connected by an edge if they share an edge on their boundaries. Moreover, two distinct regions are connected by a bold (resp. thin) edge if they share an edge e and the the third region which has e on its boundary is α_1 (resp. α_2).

The proof of the following lemma is identical to that of Lemma 2.3.

Lemma 2.7. *For any admissible coloring* $\eta \in \mathrm{Adm}_0(P_{1,1;m})$ $(m \in \mathbb{N})$*, we have* $(\eta(\alpha_1),\eta(\alpha_2)) = (0,0)$ *or* $(1,1)$.

Lemma 2.8. *Let* $\eta \in \mathrm{Adm}_0(P_{1,1;m})$ $(m \in \mathbb{N})$ *satisfy* $(\eta(\alpha_1),\eta(\alpha_2)) = (0,0)$. *Let* c_1, c_2 *be two regions of* $R(G_{1,1;m})$ *which are connected by an edge in* $G_{1,1;m}$. *If* $\eta(c_1) = 0$, *then* $\eta(c_2) = 0$.

Proof. Let e be an edge where the two regions c_1 and c_2 share their boundary. Then the third region c_3 which has e on its boundary is α_1 or α_2 (see Fig. 11). Since $\eta(\alpha_1) = \eta(\alpha_2) = 0$, and the triple $(\eta(c_1),\eta(c_2),\eta(c_3) = 0)$ is admissible, $\eta(c_2)$ must be 0 if $\eta(c_1) = 0$. □

Lemma 2.9. *If* $\eta \in \mathrm{Adm}_0(P_{1,1;m})$ $(m \in \mathbb{N})$ *satisfies* $(\eta(\alpha_1),\eta(\alpha_2)) = (0,0)$, *then we have* $\eta(c) = 0$, *for any* $c \in R(P_{1,1;m})$.

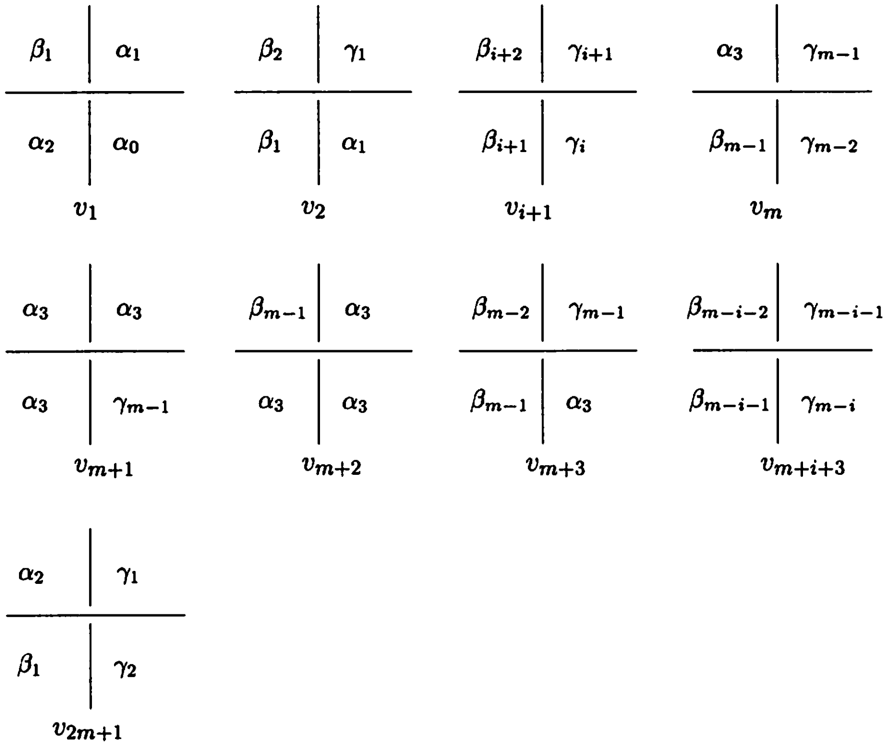

FIGURE 11. The vertices of $P_{1,1,m}$, $i = 1, 2, \ldots, m-3$.

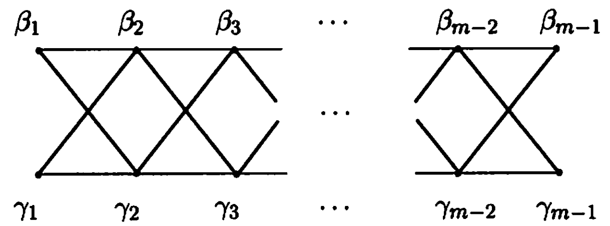

FIGURE 12. The graph $G_{1,1;m}$.

Proof. Focus on the vertex v_1 Fig. 11. Then we see that the three regions α_1, α_1 (appearing twice) and β_1 share the boundary on an edge. This implies that $\eta(\beta_1) = 0$ since the triple

$$(\eta(\alpha_1), \eta(\alpha_1), \eta(\beta_1)) = (0, 0, \eta(\beta_1))$$

is admissible.

Now, the lemma follows from Lemma 2.8 and the fact the the graph $G_{1,1;m}$ is connected. □

The proof of the following lemma is identical to that of Lemma 2.4.

Lemma 2.10. *For any admissible coloring* $\eta \in \mathrm{Adm}_i(P_{1,1;m})$ $(m \in \mathbb{N},\ i = 0, 1)$ *such that* $(\eta(\alpha_1), \eta(\alpha_2)) = (1, 1)$, *we have that* $\eta(\alpha_3) = 1$.

Lemma 2.11. *For any admissible coloring* $\eta \in \mathrm{Adm}_1(P_{1,1;m})$ $(m \in \mathbb{N})$, *we have* $(\eta(\alpha_1), \eta(\alpha_2)) = (1, 1)$.

Proof. Suppose that $\eta(\alpha_1) = 0$. Consider the coloring around the vertex v_1. Then we see that there exists an edge for which the germs of faces consist of α_1, α_1 (appearing twice) and β_1. This implies that $\eta(\beta_1) = 0$ since the triple

$$(\eta(\alpha_1), \eta(\alpha_1), \eta(\beta_1)) = (0, 0, \eta(\beta_1))$$

is admissible. Therefore, the colors of regions which are connected by a path of bold edges with β_1 in $G_{1,1;m}$ are all 0 and it follows that $\eta(\beta_{m-1}) = 0$ or $\eta(\gamma_{m-1}) = 0$, depending on whether m is even or odd. Focusing on the colorings around the vertices v_{m+1} and v_{m+2}, we see that both the triples $(\eta(\alpha_1), \eta(\beta_{m-1}), \eta(\alpha_3))$ and $(\eta(\alpha_1), \eta(\gamma_{m-1}), \eta(\alpha_3))$ are admissible. This implies that $\eta(\alpha_3) = 0$ since $(0, 0, 1)$ is not admissible. Looking at the coloring around the vertex v_{m+1}, we have that

$$(\eta(\alpha_2), \eta(\alpha_3), \eta(\alpha_3)) = (\eta(\alpha_2), 0, 0)$$

is admissible and hence $\eta(\alpha_2) = 0$. On the other hand looking at the coloring around the vertex v_1, we have that

$$(\eta(\alpha_0), \eta(\alpha_1), \eta(\alpha_2)) = (1, 0, \eta(\alpha_2))$$

is admissible. This is contradiction.

Suppose $\eta(\alpha_2) = 0$. The argument is similar to the previous one. Consider the coloring around the vertex v_1. Then we see that there exists an edge for which the germs of faces consist of α_2, α_2 (appearing twice) and β_1. This implies that $\eta(\beta_1) = 0$ since the triple

$$(\eta(\alpha_2), \eta(\alpha_2), \eta(\beta_1)) = (0, 0, \eta(\beta_1))$$

is admissible. Therefore, the colors of regions which are connected by a path of thin edges with β_1 in $G_{1,1;m}$ are all 0 and it follows that $\eta(\beta_{m-1}) = 0$. Focusing on the coloring around the vertex v_{m+1}, we see that the triple $(\eta(\alpha_1), \eta(\beta_{m-1}), \eta(\alpha_3))$ is admissible. This implies that $\eta(\alpha_3) = 0$ since $(0, 0, 1)$ is not admissible. Looking at the coloring around the vertex v_{m+1}, we have that

$$(\eta(\alpha_1), \eta(\alpha_3), \eta(\alpha_3)) = (\eta(\alpha_1), 0, 0)$$

is admissible and hence $\eta(\alpha_1) = 0$. On the other hand looking at the coloring around the vertex v_1, we have that

$$(\eta(\alpha_0), \eta(\alpha_1), \eta(\alpha_2)) = (1, \eta(\alpha_1), 0)$$

is admissible. This is contradiction.

This completes the proof. □

Hereinafter, we sometimes use $c = 0$ or 1, in place of $\eta(c) = 0$ or 1, to imply that the coloring of the region c is 0 or 1 by abuse of notation.

We define some notation needed later. Assume that there exists a special polyhedron P'_m (resp. P''_m) the neighborhoods of whose vertices are shown in Figs. 13, 14 and 15 (resp. Figs. 16, 17 and 18). (Recall that we use the presentation shown in Fig. 10.) We also assume that the coloring of the three distinguished faces α_1 α_2 and α_3 are fixed to be 1. Hence the faces which have not yet been colored are $\beta_1, \beta_2, \ldots, \beta_{m-1}$ and

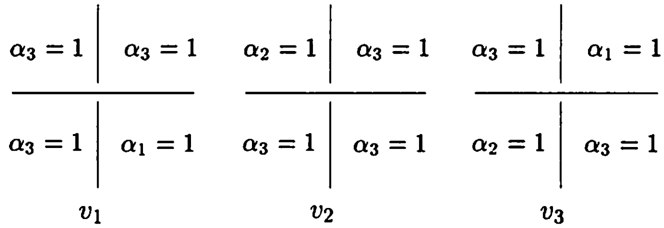

FIGURE 13. P'_1.

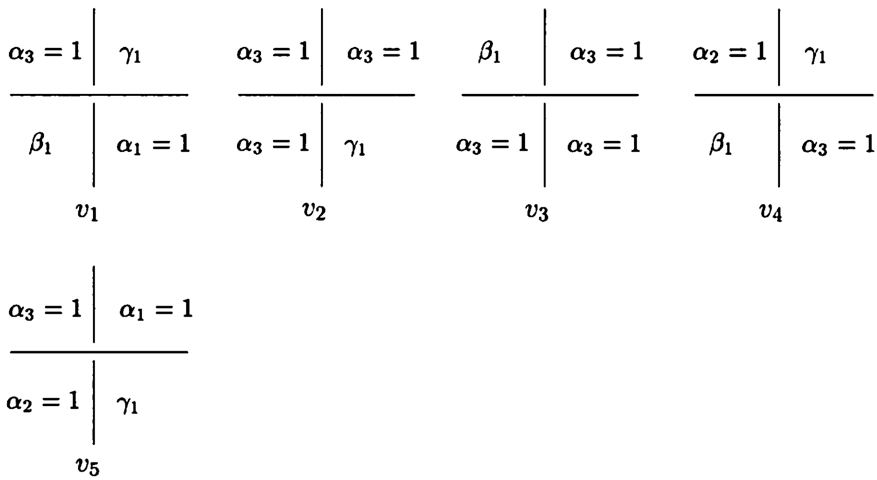

FIGURE 14. P'_2.

$\gamma_1, \gamma_2, \ldots, \gamma_{m-1}$. One may consider whether there exist such spines P'_m and P''_m with the designated neighborhoods of the vertices shown in Figs. 13–18. However, we only consider the neighborhoods of all vertices, and we are not interested in the realizability of them as special polyhedrons embedded in a 3-manifold. For this reason, we should say that P'_m and P''_m are "virtual" spines. It is possible to consider the colorings and state-sums for them and this fact is enough for a later proof.

Denote by $\mathrm{adm}(P'_m)$ (resp. $\mathrm{adm}(P''_m)$) the set of all admissible colorings

$$\eta : \{\beta_i, \gamma_i \mid i = 1, 2, \ldots, m-1\} \to \{0, 1\}$$

of P'_m (resp. P''_m). Set $q_m(\varepsilon) := \sum_{\eta \in \mathrm{adm}(P'_m)} \omega(\eta)$ (resp. $r_m(\varepsilon) := \sum_{\eta \in \mathrm{adm}(P''_m)} \omega(\eta)$).

The following two lemmas follow from direct calculation as in Section 2.3.

Lemma 2.12. $q_1(\varepsilon) = -2\varepsilon + 3$ *and* $q_2(\varepsilon) = -3\varepsilon + 5$.

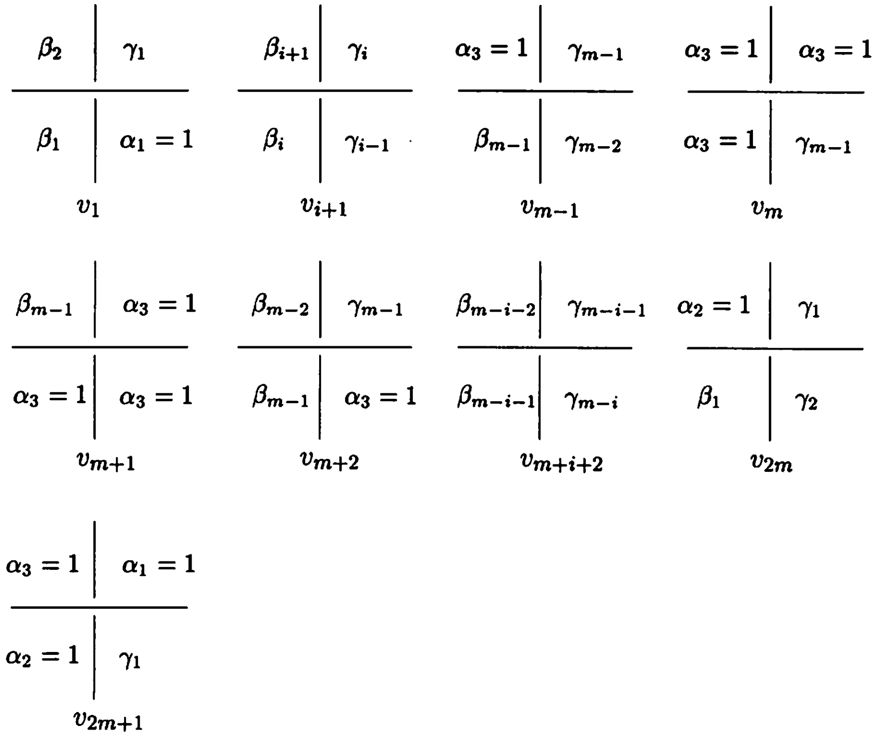

FIGURE 15. P'_m, $m \geqslant 3$, $i = 1, 2, \ldots, m-3$.

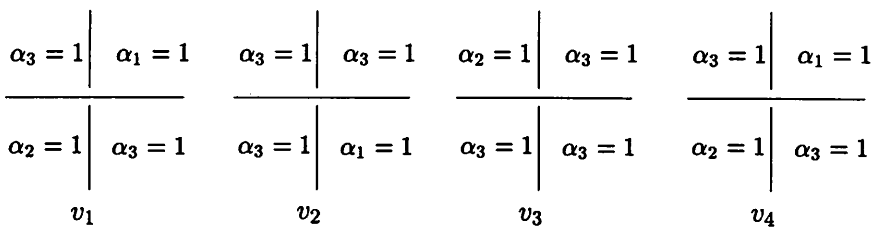

FIGURE 16. P''_1.

Lemma 2.13. *$r_1(\varepsilon) = 5\varepsilon - 8$ and $r_2(\varepsilon) = 9\varepsilon - 14$.*

Proof of Lemma 2.6. We first consider the $\mathcal{I}_0^\varepsilon$ invariant. By Section 2.3 the theorem is valid for $m = 1$ and 2.

Let $m \geqslant 3$ be an integer. Set $p_m(\varepsilon) := \mathcal{I}_0^\varepsilon(K_{1,1;m}) - 1$.

β_1	$\alpha_1 = 1$
$\alpha_2 = 1$	$\alpha_3 = 1$

v_1

$\alpha_3 = 1$	γ_1
β_1	$\alpha_1 = 1$

v_2

$\alpha_3 = 1$	$\alpha_3 = 1$
$\alpha_3 = 1$	γ_1

v_3

β_1	$\alpha_3 = 1$
$\alpha_3 = 1$	$\alpha_3 = 1$

v_4

$\alpha_2 = 1$	γ_1
β_1	$\alpha_3 = 1$

v_5

$\alpha_3 = 1$	$\alpha_1 = 1$
$\alpha_2 = 1$	γ_1

v_6

FIGURE 17. P_2''.

β_1	$\alpha_1 = 1$
$\alpha_2 = 1$	$\alpha_3 = 1$

v_1

β_2	γ_1
β_1	$\alpha_1 = 1$

v_2

β_{i+2}	γ_{i+1}
β_{i+1}	γ_i

v_{i+2}

$\alpha_3 = 1$	γ_{m-1}
β_{m-1}	γ_{m-2}

v_m

$\alpha_3 = 1$	$\alpha_3 = 1$
$\alpha_3 = 1$	γ_{m-1}

v_{m+1}

β_{m-1}	$\alpha_3 = 1$
$\alpha_3 = 1$	$\alpha_3 = 1$

v_{m+2}

β_{m-2}	γ_{m-1}
β_{m-1}	$\alpha_3 = 1$

v_{m+3}

β_{m-i-2}	γ_{m-i-1}
β_{m-i-1}	γ_{m-i}

v_{m+i+3}

$\alpha_2 = 1$	γ_1
β_1	γ_2

v_{2m+1}

$\alpha_3 = 1$	$\alpha_1 = 1$
$\alpha_2 = 1$	γ_1

v_{2m+2}

FIGURE 18. P_m'', $m \geqslant 3$, $i = 1, 2, \ldots m-3$.

Let $\eta_0 \in \mathrm{Adm}_0(P_{1,1;m})$ be the coloring such that $\eta_0(c) = 0$ for any $c \in R(P_{1,1;m})$. By Lemmas 2.7 and 2.9, the colored Turaev-Viro invariant $\mathcal{I}_0^\varepsilon(K_{1,1;m})$ can be described as

$$\mathcal{I}_0^\varepsilon(K_{1,1;m}) = 1 + \sum_{\eta \in \mathrm{Adm}_0(P_{1,1;m}) \setminus \{\eta_0\}} \omega(\eta),$$

and any $\eta \in \mathrm{Adm}_0(P_{1,1;m}) \setminus \{\eta_0\}$ satisfies $\eta(\alpha_1) = \eta(\alpha_2) = \eta(\alpha_3) = 1$ due to Lemma 2.10.

We divide $\mathrm{Adm}_0(P_{1,1;m}) \setminus \{\eta_0\}$ into four subsets as

$$\mathrm{Adm}_0(P_{1,1;m}) \setminus \{\eta_0\} = \mathcal{A}_{0,0} \sqcup \mathcal{A}_{1,0} \sqcup \mathcal{A}_{0,1} \sqcup \mathcal{A}_{1,1},$$

where $\mathcal{A}_{i,j} = \{\eta \in \mathrm{Adm}_0(P_{1,1;m}) \setminus \{\eta_0\} \mid \eta(\beta_1) = i, \eta(\gamma_1) = j\}$.

We first consider the set $\mathcal{A}_{0,0}$ Since the colors of β_1 and γ_1 are 0, it follows that the colors of β_2 and γ_2 are 1 for any admissible coloring (recall the graph $G_{1,1;m}$ and the definition of admissibility).

The coloring around the vertices of $P_{1,1;m}$ can be described as in Fig. 19. For any $\eta \in \mathcal{A}_{0,0}$, we have $w_{\eta(v_1)} = w_{\eta(v_2)} = w_{\eta(v_{2m})} = w_{\eta(v_{2m+1})} = \varepsilon^{-1}$.

Now, the contribution of the vertices $v_3, v_4, \ldots, v_{2m-1}$ and the faces $\beta_3, \beta_4, \ldots, \beta_{m-1}$, $\gamma_3, \gamma_4, \ldots, \gamma_{m-1}$ for the state-sum $\mathcal{I}_0^\varepsilon$ is equal to $p_{m-2}(\varepsilon)$ (recall that $\begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}_\varepsilon = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}_\varepsilon = \varepsilon^{-1}$) and hence

$$\begin{aligned}
\sum_{\eta \in \mathcal{A}_{0,0}} \omega(\eta) &= p_{m-2}(\varepsilon) \left(\prod_{i=1,2,2m,2m+1} s_\eta(v_i) \right) w_{\eta(\beta_1)} \cdot w_{\eta(\beta_2)} \cdot w_{\eta(\gamma_1)} w_{\eta(\gamma_2)} \\
&= p_{m-2}(\varepsilon) \cdot \varepsilon^{-4} \cdot \varepsilon^2 \\
&= \varepsilon^{-2} p_{m-2}(\varepsilon) \\
&= (-\varepsilon + 2) p_{m-2}(\varepsilon).
\end{aligned}$$

Next, consider the set $\mathcal{A}_{1,0}$. Since the color of γ_1 is 0, it follows that the colors of β_2 and γ_2 are 1 for any admissible coloring. Then the coloring around the vertices of $P_{1,1;m}$ is described as in Fig. 20. For any $\eta \in \mathcal{A}_{0,0}$ we then have $w_{\eta(v_1)} = w_{\eta(v_2)} = w_{\eta(v_3)} = w_{\eta(v_{2m+1})} = \varepsilon^{-1}$.

The contribution of the vertices $v_4, v_5, \ldots, v_{2m}$ and the faces $\beta_3, \beta_4, \ldots, \beta_{m-1}$,$\gamma_3, \gamma_4, \ldots, \gamma_{m-1}$ for the state-sum $\mathcal{I}_0^\varepsilon$ equals $q_{m-2}(\varepsilon)$ and hence

$$\begin{aligned}
\sum_{\eta \in \mathcal{A}_{1,0}} \omega(\eta) &= q_{m-2}(\varepsilon) \left(\prod_{i=1,2,3,2m+1} s_\eta(v_i) \right) w_{\eta(\beta_1)} \cdot w_{\eta(\beta_2)} \cdot w_{\eta(\gamma_1)} \cdot w_{\eta(\gamma_2)} \\
&= q_{m-2}(\varepsilon) \cdot \varepsilon^{-4} \cdot \varepsilon^3 \\
&= \varepsilon^{-1} q_{m-2}(\varepsilon) \\
&= (\varepsilon - 1) q_{m-2}(\varepsilon).
\end{aligned}$$

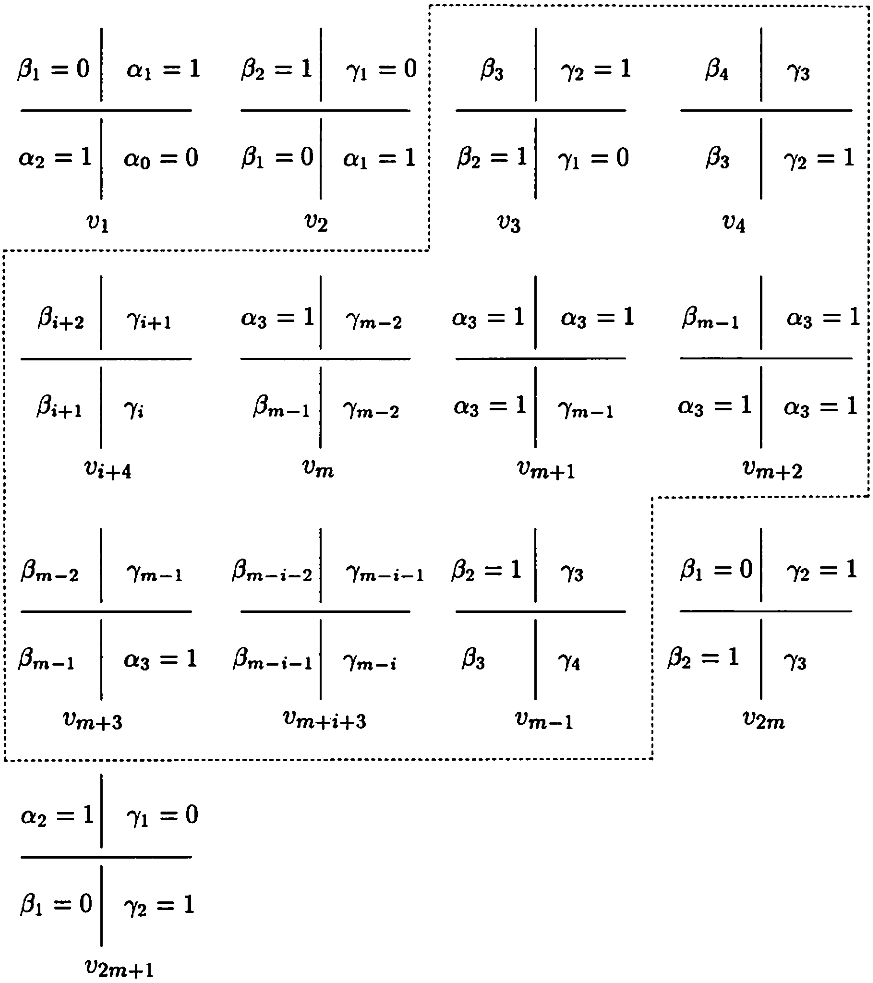

FIGURE 19. $P_{1,1,m}$ with $\alpha_0 = \beta_1 = \gamma_1 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \beta_2 = \gamma_2 = 1$, $i = 1, 2, \dots, m-5$.

We can prove following the same argument that

$$\sum_{\eta \in \mathcal{A}_{0,1}} \omega(\eta) = \varepsilon^{-1} q_{m-2}(\varepsilon).$$

Finally, consider the set $\mathcal{A}_{1,1}$ Since both colors of β_1 and γ_1 are 1, the colors of β_2 and γ_2 can vary. Let m be odd. Then the coloring around the vertices of $P_{1,1;m}$ is described as in Fig. 21. For any $\eta \in \mathcal{A}_{1,1}$, we have $w_{\eta(v_1)} = \varepsilon^{-1}$.

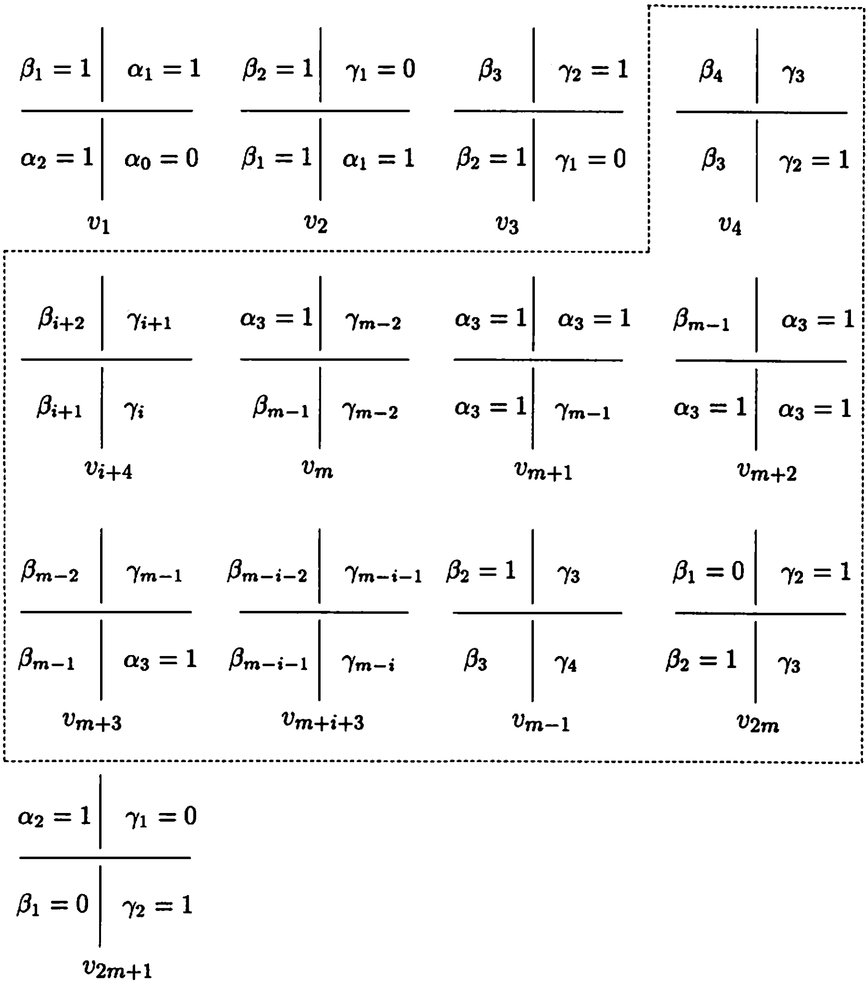

FIGURE 20. $P_{1,1,m}$ with $\alpha_0 = \gamma_1 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \gamma_2 = 1$, $i = 1, 2, \ldots, m-5$.

The contribution of the vertices $v_2, v_3, \ldots, v_{2m+1}$ and the faces $\beta_2, \beta_3, \ldots, \beta_{m-1}$,$\gamma_2, \gamma_3, \ldots, \gamma_{m-1}$ for the state-sum $\mathcal{I}_0^\varepsilon$ equals $r_{m-1}(\varepsilon)$ and hence

$$\begin{aligned} \sum_{\eta \in \mathcal{A}_{1,1}} \omega(\eta) &= r_{m-1}(\varepsilon) \cdot s_\eta(v_1) \cdot w_{\eta(\beta_1)} \cdot w_{\eta(\gamma_1)} \\ &= r_{m-1}(\varepsilon) \cdot \varepsilon^{-1} \cdot \varepsilon^2 \\ &= \varepsilon r_{m-1}(\varepsilon). \end{aligned}$$

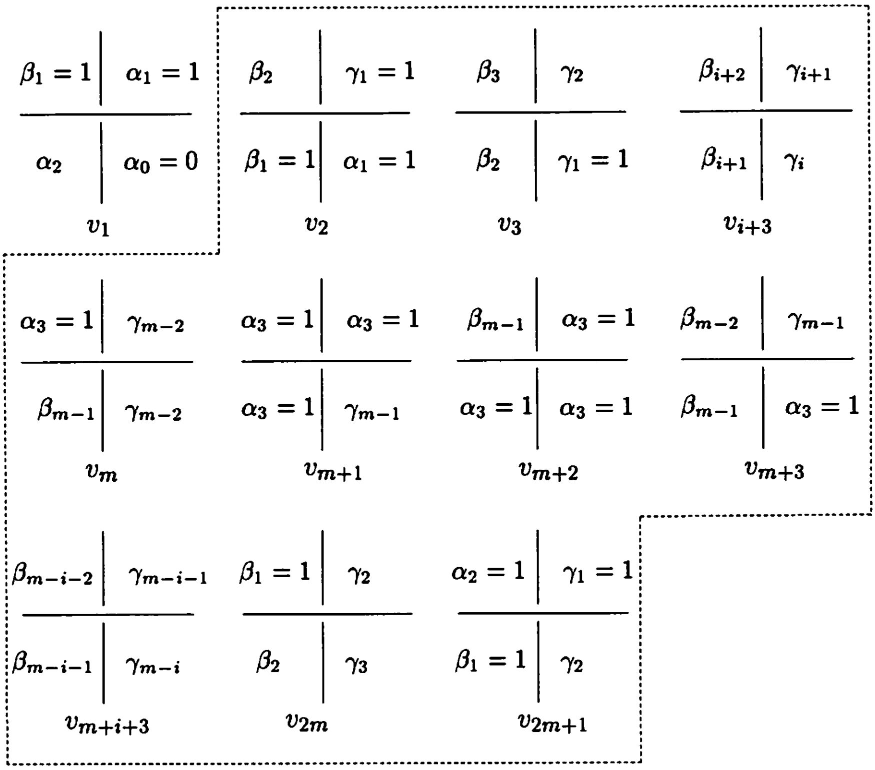

FIGURE 21. $P_{1,1,m}$ with $\alpha_0 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \gamma_1 = 1$, $i = 1, 2, \ldots, m-4$.

Therefore we have

$$\begin{aligned} p_m(\varepsilon) &= \mathcal{I}_0^\varepsilon(K_{1,1;m}) - 1 \\ &= \left(\omega(\eta_0) + \sum_{\eta \in \mathrm{Adm}_0(P_{1,1;m}) \setminus \{\eta_0\}} \omega(\eta) \right) - 1 \\ &= \sum_{\eta \in \mathcal{A}_{0,0}} \omega(\eta) + \sum_{\eta \in \mathcal{A}_{1,0}} \omega(\eta) + \sum_{\eta \in \mathcal{A}_{0,1}} \omega(\eta) + \sum_{\eta \in \mathcal{A}_{1,1}} \omega(\eta) \\ &= (-\varepsilon + 2) p_{m-2}(\varepsilon) + 2(\varepsilon - 1) q_{m-2}(\varepsilon) + \varepsilon r_{m-1}(\varepsilon). \end{aligned}$$

This proves the first relation in the recurrence relations (1). We can prove the same equation for even m in the same argument.

For the second and the third relations, we can apply essentially the same argument. This completes the statement for $\mathcal{I}_0^\varepsilon$.

Next, we consider the invariant $\mathcal{I}_1^\varepsilon$. By Lemma 2.10, any $\eta \in \mathrm{Adm}_1(P_{1,1,m})$ satisfies $\eta(\alpha_1) = \eta(\alpha_2) = 1$ and thus $\eta(\alpha_3) = 1$ by Lemma 2.10.

Now, the coloring around the vertices can be described as in Fig. 22. It follows that

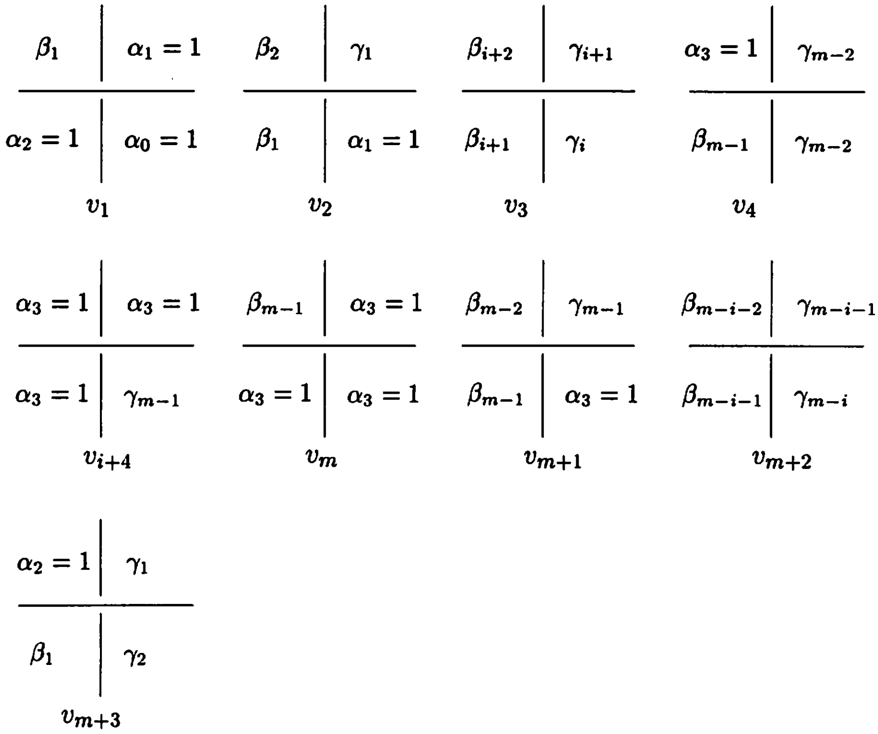

FIGURE 22. $P_{1,1;m}$ with $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 1$.

the situation is almost the same as in the spine P'_m. The only difference is that the o-spine $P_{1,1;m}$ has an extra region α_0 whose color is 0. Therefore we have

$$\begin{aligned}\mathcal{I}^{\varepsilon}_1(K_{1,1;m}) &= \sum_{\eta \in \mathrm{Adm}_1(P_{1,1;m})} \omega(\eta) \\ &= w_{\eta(\alpha_0)} \cdot q_m(\varepsilon) \\ &= \varepsilon q_m(\varepsilon),\end{aligned}$$

whence the conclusion.

Example. By Lemma 2.6, we have

$$\begin{aligned} p_3(\varepsilon) &= (-\varepsilon+2)p_1(\varepsilon) + 2(\varepsilon-1)q_1(\varepsilon) + \varepsilon r_2(\varepsilon) \\ &= (-\varepsilon+2)(-\varepsilon+2) + 2(\varepsilon-1)(-2\varepsilon+3) + \varepsilon(9\varepsilon-14) \\ &= 6\varepsilon^2 - 8\varepsilon - 2, \\ &= -2\varepsilon + 4, \\ q_3(\varepsilon) &= (-\varepsilon+2)(-\varepsilon+2) + (2\varepsilon-3)(-2\varepsilon+3) - (9\varepsilon-14) \\ &= -4\varepsilon + 6. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \mathcal{I}_0^{\varepsilon}(K_{1,1;3}) &= p_3(\varepsilon) + 1 = -2\varepsilon + 5, \\ \mathcal{I}_1^{\varepsilon}(K_{1,1;3}) &= \varepsilon q_3(\varepsilon) = 2\varepsilon - 4. \end{aligned}$$

Let M be a compact 3-manifold. We denote by $t(M)$ the *ε-invariant* of M.

Lemma 2.14. *For any $m \in \mathbb{N}$, we have*

$$\mathcal{I}_0^{\varepsilon}(K_{1,1;m}) + \mathcal{I}_1^{\varepsilon}(K_{1,1;m}) = 1$$

Proof. Since we have

$$\mathcal{I}_0^{\varepsilon}(K_{1,1;m}) + \mathcal{I}_1^{\varepsilon}(K_{1,1;m}) = t(S^3),$$

by definition, and $t(S^3) = 1$ (see e.g. [11]), the lemma follows. □

Lemma 2.15. *For any $m \in \mathbb{N}$, we have $p_m(\varepsilon) = -\varepsilon q_m(\varepsilon)$.*

Proof. By Lemma 2.6 and Lemma 2.14, we have

$$\mathcal{I}_0^{\varepsilon}(K_{1,1;m}) + \mathcal{I}_1^{\varepsilon}(K_{1,1;m}) = 1 + p_m(\varepsilon) + \varepsilon q_m(\varepsilon) = 1,$$

for any $m \in \mathbb{N}$, whence the assertion. □

By Lemma 2.15, we can improve Lemma 2.6 as in the following final form:

Lemma 2.16. *Set*

$$p_1(\varepsilon) = -\varepsilon + 2,\ p_2(\varepsilon) = -2\varepsilon + 3,\ p_3(\varepsilon) = -2\varepsilon + 4, \tag{2}$$

and consider the following recurrence relations:

$$p_{m+3}(\varepsilon) = (-\varepsilon + 2)p_{m+2}(\varepsilon) + (\varepsilon - 2)p_{m+1}(\varepsilon) + p_m(\varepsilon)\ (m \geqslant 1), \tag{3}$$

Then we have

$$\mathcal{I}_0^{\varepsilon}(K_{1,1,m}) = p_m(\varepsilon) + 1,\ \mathcal{I}_1^{\varepsilon}(K_{1,1,m}) = -p_m(\varepsilon), \tag{4}$$

for any $m \in \mathbb{N}$.

Proof. By Lemma 2.15, we can refine the recurrence relations in Lemma 2.6 as

$$\begin{cases} p_1(\varepsilon) = -\varepsilon + 2,\ p_2(\varepsilon) = -2\varepsilon + 3, \\ r_1(\varepsilon) = 5\varepsilon - 8,\ r_2(\varepsilon) = 9\varepsilon - 14, \end{cases}$$

$$\begin{cases} p_m(\varepsilon) = (\varepsilon - 2)p_{m-2}(\varepsilon) + \varepsilon r_{m-1}(\varepsilon)\ (m \geqslant 3), \\ r_m(\varepsilon) = (-4\varepsilon + 7)p_{m-2}(\varepsilon) + (-\varepsilon + 2)r_{m-1}(\varepsilon)\ (m \geqslant 3). \end{cases} \tag{5}$$

The first relation of (5) gives

$$r_m(\varepsilon) = \varepsilon^{-1}p_{m+1}(\varepsilon) - \varepsilon^{-1}(\varepsilon - 2)p_{m-1}(\varepsilon)$$

and

$$r_{m-1}(\varepsilon) = \varepsilon^{-1}p_m(\varepsilon) - \varepsilon^{-1}(\varepsilon - 2)p_{m-2}(\varepsilon),$$

and hence the required formula is obtained by substituting $\varepsilon^{-1}p_{m+1}(\varepsilon) - \varepsilon^{-1}(\varepsilon - 2)p_{m-1}(\varepsilon)$ for $r_m(\varepsilon)$ and $\varepsilon^{-1}p_m(\varepsilon) - \varepsilon^{-1}(\varepsilon - 2)p_{m-2}(\varepsilon)$, for $r_{m-1}(\varepsilon)$ in the second relation of (5). □

Now, Theorem 2.5 is obtained from (2), (3) and (4) by an elementary calculation.

Remark. Our method does not essentially require that the knot is embedded in S^3. In fact, we can calculate colored Turaev-Viro invariants of many knots in lens spaces using o-spines arising from $(1,1)$-decompositions. However, it still remains difficulty to have analogous formulae for *infinite* series of knots $K_{p,q;m} \subset L(p,q)$, $m \in \mathbb{N}$ (see [10]) as above.

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