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# THE REIDEMEISTER-TURAEV TORSION OF STANDARD SPIN$^c$ STRUCTURES ON SEIFERT FIBERED 3-MANIFOLDS

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ABSTRACT. The Reidemeister-Turaev torsion is an invariant of 3-manifolds equipped with Spin$^c$ structures. Here, a Spin$^c$ structure of a 3-manifold is a homology class of non-singular vector fields on it. Each Seifert fibered 3-manifold has a standard Spin-c structure, which is represented as vector fields everywhere tangential to its Seifert fibration. In the present paper, using punctured Heegaard diagrams, we give an algorithm to compute the Reidemeister-Turaev torsion of a standard Spin$^c$ structure of a Seifert fibered 3-manifold starting from a given Seifert parameter.

## INTRODUCTION

Reidemeister-Turaev torsion is an invariant of 3-manifolds equipped with Spin$^c$ structures. This invariant is defined by Turaev [16] as a refinement of the Reidemeister torsion, which is one of the most well-known classical invariant of 3-manifolds. A Spin$^c$ structure can be represented as a *homology class* of non-singular vector fields on the ambient 3-manifold. On the other hand, a branched standard spine of a 3-manifold carries a non-singular vector field. The computation of the Reidemeister-Turaev torsion using branched standard spines is first introduced in [3] for the case with non-empty boundary and then in [1] for the closed case. In [9] we developed the method via Heegaard splittings compatible with the branched standard spines. In the case of closed 3-manifolds, it is exactly the Heegaard diagram equipped with a special region. In [11] we introduce Heegaard-type diagrams, which we call *punctured Heegaard diagrams*, to present branched spines and it allows to compute the Reidemeister-Turaev torsion quite easily.

In the present paper, we introduce the method for constructing punctured Heegaard diagrams of Seifert fibered 3-manifolds equipped with standard Spin$^c$ structures and then explain how to compute its Reidemeister-Turaev torsion. This the first method to compute the invariant of standard Spin$^c$ structure of Seifert fibered 3-manifolds.

*Notation.* Let $X$ be a subset of a given topological space or a manifold $Y$. Throughout this paper, we will denote the interior of $X$ by $\mathrm{Int}\,X$, the closure of $X$ by $\overline{X}$ and the number of components of $X$ by $\#X$. We will use $\eta(X;Y)$ to denote a regular neighborhood of $X$ in $Y$. If the ambient space $Y$ is clear from the context, we simply denote it by $\eta(X)$. By 3-manifold, we always mean a *connected, compact* and *oriented* one, with or without boundary, unless otherwise mentioned.

## 1. Preliminaries

**1.1. Spin$^c$ structures.** Let $M$ be a closed smooth 3-manifold. Two non-singular vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ on $M$ are said to be *homologous* if there exists a closed 3-ball $B \subset M$ such that the restrictions of $\mathcal{V}_1$ and $\mathcal{V}_2$ to $M \setminus \mathrm{Int}\, B$ are homotopic as non-singular vector fields. A *smooth Spin$^c$ structure* is a homology class of non-singular vector fields. We denote by $\mathrm{Spin}^c(M)$ the set of smooth Spin$^c$ structure on $M$. The action of $H_1(M)$ to $\mathrm{Spin}^c(M)$ is defined through Reeb surgery, see [20, 13] for details.

**1.2. The Reidemeister-Turaev torsion.** Let $F$ be a field and let $E$ be an $n$-dimensional vector space over $F$. For two ordered bases $b = (b_1, \ldots, b_n)$ and $c = (c_1, \ldots, c_n)$ of $E$, we write $[b/c] = \det(a_{ij}) \in F^\times$, where $b_i = \sum_{j=1}^n a_{ij} c_j$. The bases $b$ and $c$ are said to be *equivalent* if $[b/c] = 1$.

Let $C = (0 \xrightarrow{\partial_m} C_m \xrightarrow{\partial_{m-1}} C_{m-1} \to \cdots \to C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\partial_{-1}} 0)$ be a finite dimensional chain complex over $F$. For each $0 \le i \le m$, set $B_i = \mathrm{Im}\, \partial_i$, $Z_i = \mathrm{Ker}\, \partial_{i-1}$ and $H_i = Z_i / B_i$. The chain complex is said to be *acyclic* if $H_i = 0$ for all $i$. Suppose that $C$ is acyclic and $C_i$ is endowed with a *distinguished* basis $c_i$ for each $i$. Choose an ordered set of vectors $b_i$ in $C_i$ for each $i = 0, \ldots, m$ such that $\partial_{i-1}(b_i)$ forms a basis of $B_{i-1}$. By the above construction, $\partial_i(b_{i+1})$ and $b_i$ are combined to be a new basis $\partial_i(b_{i+1}) b_i$ of $C_i$. With this notation, the *torsion* of $C$ is defined by

$$\tau(C) := \prod_{i=0}^{m} [\partial_i(b_{i+1}) b_i / c_i]^{(-1)^{i+1}} \in F^\times.$$

Let $M$ be a compact connected orientable smooth manifold of an arbitrary dimension. Let $X$ be a CW-decomposition of $M$, $\widetilde{X} \to X$ be its maximal abelian covering and $F$ be a field. We can equip $\widetilde{X}$ with the CW-structure naturally induced by that of $X$, and then we regard $C_*(\widetilde{X})$ as a left $\mathbb{Z}[\pi_1(X, *)]$-module via the monodromy. Let $\{e_i^k\}$ be the set of all oriented $k$-cells in $X$, and $\{\widetilde{e}_i^k\}$ be a family of their lifts to $\widetilde{X}$. Endow orientation with each of these cells and order the cells $\{\widetilde{e}_i^k\}$, for each $k$, in an arbitrary way. Then this family gives an ordered $\mathbb{Z}[H_1(X)]$-basis of $C_k(\widetilde{X})$. In this way, we can regard $C_*(\widetilde{X})$ as an ordered, based chain complex.

Let $\varphi : \mathbb{Z}[\pi_1(X)] \to F$ be a ring homomorphism. If the based chain complex $C_*^\varphi(X) = F \otimes_\varphi C_*(\widetilde{X})$ over $F$ is acyclic, the *($\varphi$-twisted) Reidemeister torsion* of $X$ is

$$\tau^\varphi(M) := \tau(C_*^\varphi(X)) \in F^\times / \pm \varphi(H_1(M)).$$

Otherwise, set $\tau^\varphi(M) := 0 \in F$.

Let $M$ be a smooth 3-manifold and let $X$ be its CW-decomposition.

A family of cells of $\hat{X}$ is said to be *fundamental* if over each cell of $X$ exactly one cell of this family lies. When we choose a fundamental family $\{\hat{e}_i^k\}$ of cells of $\hat{X}$ and orient and order these cells in arbitrary way, it becomes a free $\mathbb{Z}[H_1(X)]$-basis of $C_k(\hat{X})$. (i.e. $C_k(\hat{X}) = \bigoplus_i \mathbb{Z}[H_1(X)] \hat{e}_i^k$). In this way, we can regard $C_*(\hat{X})$ as a chain complex with basis.

An important fact is that we can naturally define the fundamental family of cells for each Spin$^c$ structure. Then Turaev refined the Reidemeister torison to be an invariant $\tau^{\varphi}(M,\theta) \in F/\pm 1$ of Spin$^c$ structures on a 3-manifold $M$.

In [1, 3], the inverse of the above bijection is described via the notion of *branched standard spine.*

Let $M$ be a Seifert fibered 3-manifold. Then its fiber structure naturally defined

We call a non-singular vector field (resp. a Spin$^c$ structure) on a Seifert fibered 3-manifold is *standard* if it is everywhere tangentical to a Seifert fibration. In [15], Taniguchi, Tsuboi and Yamashita introduced an algorithm to obtain a *DS-diagram* of a standard vector field on an arbitrary closed Seifert fibered manifold starting from its Seifert data.

1.3. **Branched spines.** Let $N$ be a compact orientable 3-manifold. A branched surface $P \subset N$ is a union of finitely many compact smooth surfaces glued together to form a compact subspace locally modeled on one of the three possibilities in Fig. 1. Note that

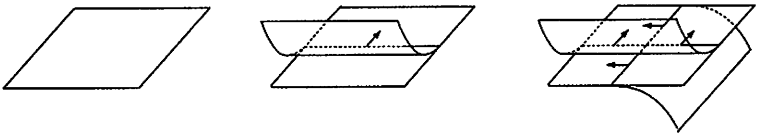

FIGURE 1. Local pictures of a branched surface.

the general definition of branched surface allows more sheets than just two on one side and one on the other side, but we only consider this situation (which is generic and stable, i.e. corresponds to an open dense set in the space of branched surfaces).

The *branch locus* $S(P)$ of $P$ is the set of points none of whose neighborhoods (in $P$) is a disk. $S(P)$ is a collection of smooth immersed curves in $P$. Let $V(P)$ be the set of double points of $S(P)$. We associate with every component of $S(P) \setminus V(P)$ a normal vector (in $P$) pointing in the locally one-sheeted direction, as shown in Fig. 1. We call a component of $P \setminus S(P)$ a *sector* of $P$. Let $R$ be a sector of $P$. If all branch directions along $\partial\overline{R}$ point out from $R$, then $P \setminus R$ is still a branched surface, see Fig. 2 (i). One can

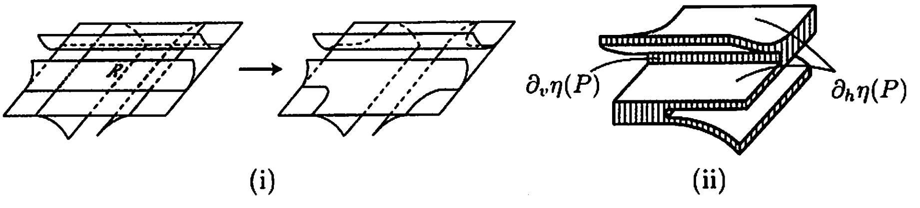


FIGURE 2. (i) Removable sector; (ii) A regular neighborhood of a branched surface.

regard $\eta(P)$ as an interval bundle over $P$ as drawn in Fig. 2 (ii). The boundary $\partial\eta(P)$ decomposes into two parts: the endpoints of the fibers, $\partial_h\eta(P)$, and the rest, $\partial_v\eta(P)$. In this paper, all branched surfaces are assumed to be *transversely oriented*, that is, $P$

is equipped with a global orientation on the 1-foliation of $\eta(P)$ whose leaves are fibers of $\eta(B)$. Refer to [6, 14] for more details about branched surfaces.

A branched surface $P \subset N$ is called a *branched spine* (of $N$) if $N$ collapses onto $P$. A branched spine $P$ is naturally stratified as $V(P) \subset S(P) \subset P$. A branched spine $P$ is said to be *standard* if this stratification induces a CW decomposition of $P$, namely, there is no loop in $S(P)$ and sectors are disks. See [2] for a precise definition. If $P$ is a branched spine of a compact 3-manifold $N$ with $\partial N = S^2$, then $P$ is also called a branched spine of the closed 3-manifold $M$ obtained from $N$ by attaching a 3-ball to the unique 2-sphere boundary. A branched spine of a closed 3-manifold is called a *flow-spine* if $\partial_v \eta(P)$ is an annulus.

In [2], Benedetti and Petronio proved that every orientable 3-manifold admits a branched (standard) spine and it naturally encodes a well-defined homotopy class of vector fields, which is called the *concave traversing field*, on the ambient manifold. We require the flow intersects $P$ in the same direction as the fixed transverse orientation. In the case where $P$ is a flow-spine of a closed oriented 3-manifold $M$, one can extend the concave traversing field, whose orbits are the $I$-fibers of the regular neighborhood of the spine, to the whole of $M$.

1.4. **Punctured Heegaard diagrams.** We only consider closed orientable 3-manifolds. By a *Heegaard diagram* we means a triple $(S; \alpha, \beta)$ where $S$ is a closed, connected, orientable surface of genus $g$ and $\alpha = \bigcup_{i=1}^{g} \alpha_i$ and $\beta = \bigcup_{i=1}^{g} \beta_i$ are compact, mutually transverse 1-manifolds on $S$. We always assume that the two families of curves $\alpha$ and $\beta$ are in *general position.* A Heegaard diagram gives rise to a 3-manifold

$$M_{(S;\alpha,\beta)} := S \times [-1,1] \bigcup_{\alpha \times \{-1\}} \text{(2-handles)} \bigcup_{\beta \times \{1\}} \text{(2-handles)}$$

obtained by adding 2-handles $H_{\alpha_1}, \dots, H_{\alpha_m}$ and $H_{\beta_1}, \dots, H_{\beta_n}$ to $S \times [-1,1]$ along the curves $\alpha_1 \times \{-1\}, \dots, \alpha_m \times \{-1\}$ and $\beta_1 \times \{1\}, \dots, \beta_m \times \{1\}$, respectively. We denote by $\widehat{M}_{(S;\alpha,\beta)}$ the manifold obtained by adding 3-handles along the resulting 2-sphere boundary component of $M_{(S;\alpha,\beta)}$. We will denote the core disk of $H_{\alpha_i}$ (resp. $H_{\beta_j}$) (fairly extended so that its boundary is on $S$) by $D_{\alpha_i}$ (resp. $D_{\beta_j}$). When we consider (and draw in $\mathbb{R}^3$) a Heegaard diagram, we always equip the surface $S$ with the positive normal $\mathbf{w}_x$ ($x \in S$) pointing toward the $\alpha$ side, and with the orientation $(\mathbf{u}_x, \mathbf{v}_x)$ such that $(\mathbf{u}_x, \mathbf{v}_x, \mathbf{w}_x)$ gives the left-hand orientation on $\mathbb{R}^3$.

A Heegaard diagram is said to be *oriented* if the 1-manifolds $\alpha$ and $\beta$ are oriented. A Heegaard diagram $(S; \alpha, \beta)$ with a fixed point $b_i \in \beta_i \setminus \alpha$ for each $\beta_i$ is said to be *based.* A Heegaard diagram $(S; \alpha, \beta)$ is said to be *standard* if every connected component of $S \setminus (\alpha \cup \beta)$ is an open ball. It is clear that we can make any Heegaard diagram standard up to isotopy of $\beta$. We often denote an oriented, based Heegaard diagram by $(S; \vec{\alpha}, \vec{\beta}, \{b_k\}_{k=1}^{g})$. A system of pairwise disjoint, simple, closed, oriented curves $\gamma = \bigcup_{i=1}^{g} \gamma_i$ on $S$ is called a *dual system* of $\beta$ if each $\gamma_i$ intersects $\beta_i$ transversely once at the point $b_i$ in the positive shown in Fig. 3, where $(u_x, v_x)$ is compatible of the fixed orientation of $S$, and $\gamma_i \cap \beta_j = \emptyset$ when $i \neq j$.

Given a Heegaard diagram $(S; \alpha, \beta)$, let $D$ be a disk component of $S \setminus (\alpha \cup \beta)$. Then $D$ is said to be *joining* if it satisfies the following: i) $\partial \overline{D}$ is a disjoint union of simple

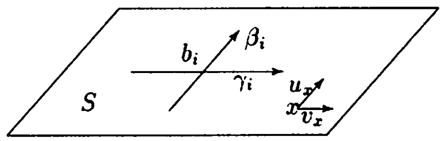


FIGURE 3. The positive intersection with a dual loop.

closed curves, where the closure is taken in the surface $S$; and ii) $\partial\overline{D}\cap\alpha_j$ (resp. $\partial\overline{D}\cap\beta_j$) is a connected arc. A punctured Heegaard diagram $(S;\alpha,\beta;D)$ is said to be *standard* if every connected component of $S\setminus(\alpha\cup\beta\cup D)$ is an open ball.

We call a Heegaard diagram $(S;\alpha,\beta)$ with joining union $J$ a *punctured Heegaard diagram* and denote it by $(S;\alpha,\beta;J)$. A punctured Heegaard diagram gives rise to a 3-manifold $M_{(S;\alpha,\beta;J)}$ obtained by adding 2-handles $H_{\alpha_1},\ldots,H_{\alpha_g}$ and $H_{\beta_1},\ldots,H_{\beta_g}$ to $\overline{S\setminus J}\times[-1,1]$ along the curves $\alpha_1\times\{-1\},\ldots,\alpha_g\times\{-1\}$ and $\beta_1\times\{1\},\ldots,\beta_g\times\{1\}$, respectively. We denote by $\widehat{M}_{(S;\alpha,\beta;J)}$ the manifold obtained by adding a 3-handle along all resulting 2-sphere boundary components of $M_{(S;\alpha,\beta)}$.

**Lemma 1.1.** *Let $(S;\alpha,\beta)$ be a Heegaard diagram such that $\widehat{M}_{(S;\alpha,\beta)}$ is a closed 3-manifold and $\#\alpha=\#\beta$ equals to the genus of $S$. Assume that $(S;\alpha,\beta)$ has a joining disk $D$. Then we have $\widehat{M}_{(S;\alpha,\beta)}\cong\widehat{M}_{(S;\alpha,\beta;D)}$.*

Let $(S;\vec{\alpha},\vec{\beta};\{b_k\}_{k=1}^{g})$ be an oriented, based Heegaard diagram of genus $g$ such that $M:=\widehat{M}_{(S;\alpha,\beta)}$ is closed.

Let $p$ be a point on $\alpha_i$. Then the normal vector $\mathbf{u}_p$ of $\alpha_i$ on $p$ is defined so that $(\mathbf{u}_p,\mathbf{v}_p)$ is coherent to the fixed orientation of $S$, where $\mathbf{v}_p\in T_p\alpha_i$ is defined by the orientation of $\alpha_i$.

Then $\alpha_i$ determines an element $x_i\in\pi_1(M,*)$ and $\beta_j$ determines $r_j=r_j(x_1,\ldots,x_g)\in\pi_1(M,*)$ starting at the point $b_j$ and following the oriented loop $\beta_j$, for each $i,j=1,\ldots,g$. Here we use the convention such that at each point $p\in\alpha_i\cap\beta_j$ we read $x_i$ when $\mathbf{v}_p$ is coherent to the orientation of $\beta_j$. Moreover, if we choose a dual system $\gamma=\bigcup_{i=1}^{g}\gamma_i$ of $\beta$, $\gamma_i$ determines $y_j\in\pi_1(M,*)$ in the same manner. Let $p:\mathbb{Z}[\pi_1(M,*)]\to\mathbb{Z}[H_1(M)]$ be the canonical projection and denote $[z]=p(z)$ for $z\in\pi_1(M,*)$. The following is immediate from the above setting and definition of the Reidemeister-Turaev torsion.

**Corollary 1.2.** *Let the twisted chain complex $C_*^{\varphi}(M)$ be acyclic. Then there exist two integers $k,l\in\{1,\ldots,n\}$ and $[\mathcal{V}_{(S;\vec{\alpha},\vec{\beta};\{b_j\})}]\in\mathrm{Spin}^c(M)$ such that*

$$\tau^{\varphi}(M,[\mathcal{V}_{(S;\vec{\alpha},\vec{\beta};\{b_j\})}])=\frac{\det B_{k,l}}{(\varphi([x_k])-1)(\varphi([y_l])-1)}\in F^{\times}/\pm1,$$

*where $B_{k,l}$ is the $(k,l)$-minor of the matrix $\left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1\le i,j\le g}$, namely the matrix obtained by removing $k$-th row and $l$-th column from the matrix $\left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1\le i,j\le g}$. Here, if $B_{k,l}=\emptyset$, we set $\det B_{k,l}=1$ expediently.*

We call this value the *Reidemeister-Turaev torsion* of the oriented, based Heegaard diagram $(S; \vec{\alpha}, \vec{\beta}; \{b_k\}_{k=1}^{g})$.

1.5. **BW-decompositions and DS-diagrams.** Let $P$ be a flow-spine of a closed 3-manifold $M$. Let $N$ be a regular neighborhood of $P$. Recall that $\partial N \cong S^2$. Then the collapsing $N \searrow P$ induced a retraction $\pi$ such that $N$ is the mapping cylinder of $\pi|_{\partial N} : \partial N \to P$. This map satisfies the following:

(1) $\pi^{-1}(S(P)) \cap \partial N$ is a trivalent graph;
(2) For $x \in P$, $\phi^{-1}(x)$ consists of 2, 3 or 4 points according as $x \in P \setminus S(P)$, $x \in S(P) \setminus V(P)$ or $x \in V(P)$; and
(3) There exists a circle $e$ in $\pi^{-1}(S(P)) \cap \partial N$ such that
  (a) $\partial N \setminus e$ is the disjoint union of $B$ and $W$ (this is called a *Black and White* (or simply *B-W*) *decomposition*);
  (b) Every component of $e$ has $B$ on one side and $W$ on the other side;
  (c) $\pi$ maps $e \setminus \pi^{-1}(V(P))$ bijectively onto $S(P) \setminus V(P)$; and
  (d) $\pi$ maps $B$ (resp. $W$) bijectively onto $P$.

The left-hand side of Fig. 4 depicts the B-W decomposition of $\partial N$. In the figure, the arrows shows the concave traversing field on $N$ defined by the branched spine $P$. Remark that the curve $e$ consists of the concave points on to the boundary. The right-hand side shows the trivalent graph $\pi^{-1}(S(P)) \cap \partial N$. In the figure, the arrows shows the retraction $\pi$ induced by the collapsing See [2, Section 3.3] for more details on B-W decomposition.

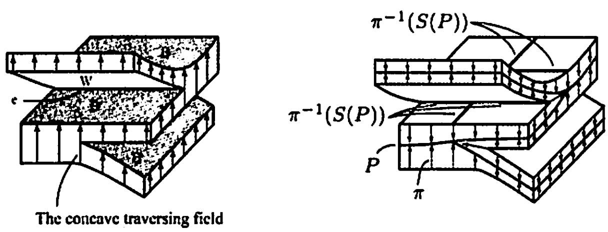


FIGURE 4. The B-W decomposition of $\partial N$.

The above description provides a way to present the flow-spine $P$ by a 3-regular graph $G := \pi^{-1}(S(P)) \cap \partial N \subset \partial N \cong S^2$ and the paring on $S^2$ given by $\pi$. This presentation is called a *DS-diagram*.

## 2. PUNCTURED HEEGAARD DIAGRAMS OF STANDARD SPIN$^c$ STRUCTURES

2.1. **construction.** Any Seifert fibered 3-manifold is constructed by gluing pieces each of which is homeomorphic to either $(S^2 \setminus \bigsqcup_{i=1}^{3} \mathrm{Int}D_i) \times S^1$, $((S^1 \times S^1) \setminus \bigsqcup_{i=1}^{3} \mathrm{Int}D_i) \times S^1$ or a fibered torus, where $D_1$, $D_2$ and $D_3$ are mutually disjoint closed disk in the surface. Our aim is to construct a punctured Heegaard diagram by gluing the diagrams corresponding to the pieces.

Let $H_R$, $H_L$ and $H_C$ be the pieces of a punctured Heegaard diagram shown in Fig. 5. For $H_R$ or $H_L$, the disks $D^-$ and $D^+$ are identified to be a meridian disk $D$ of genus 1 compact orientable surface with two boundary components.

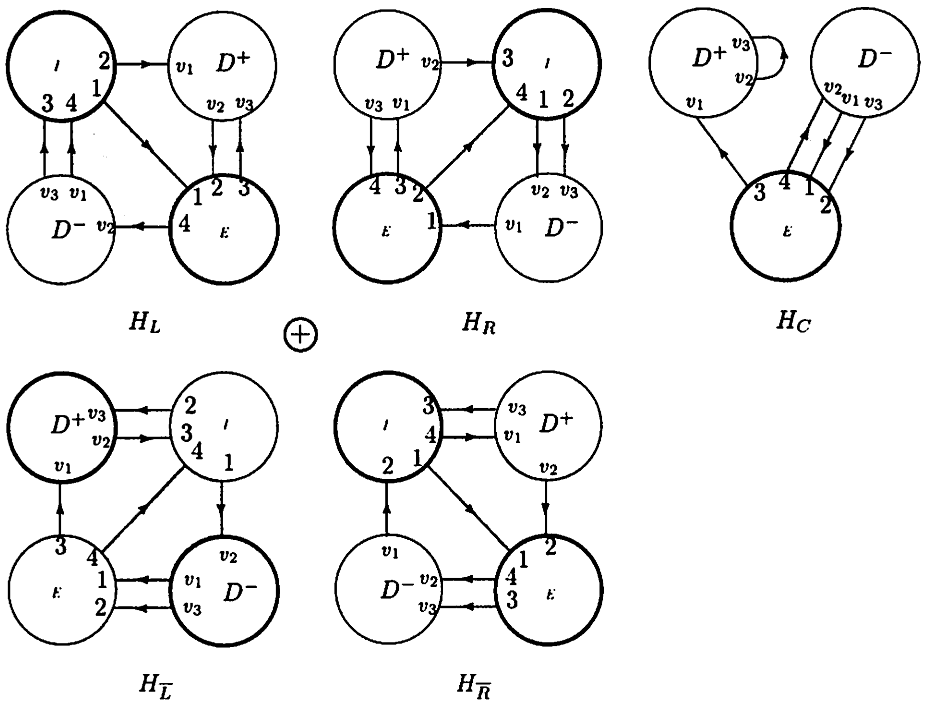


FIGURE 5. The pieces $H_L$, $H_R$, $H_{\overline{L}}$, $H_{\overline{R}}$ and $H_C$.

We use the following notation for the expansion into continuous fraction:

$$[a_1, a_2, \ldots, a_n] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

For a pair of mutually coprime natural numbers $p, q$ such that $p > q$, the word $A(p,q)$ is defined as follows:

$$A(p,q) := \begin{cases} L^{a_1} R^{a_2} L^{a_3} \cdots L^{a_{n-2}} R^{a_{n-1}} L^{a_n} & \text{(if } n \text{ is odd)} \\ L^{a_1} R^{a_2} L^{a_3} \cdots R^{a_{n-2}} L^{a_{n-1}} R^{a_n} & \text{(if } n \text{ is even),} \end{cases}$$

where $a_1, a_2, \ldots, a_n$ are natural numbers with $q/p = [a_1, a_2, \ldots, a_n, 1]$.

Given a word $A(p,q)$, where $q/p = [a_1, a_2, \ldots, a_n, 1]$, we construct a piece of punctured Heegaard diagram $H_{(p,q)}$ in the following way. Take $a_1$ copies of the diagrams $H_L$. Then attach the boundary $\partial E$ of the $i$-th diagram $H_L$ and the disk $\partial I$ of the $(i+1)$-th one along their boundaries following the numbers $1, 2, 3, 4$, for each $i = 1, 2, \ldots, a_1 - 1$. For the disk $I$ of the first diagram $H_L$, attach the disk $E$ of the diagram $H_C$. Next, take $a_2$ copies of the diagrams $H_R$. Then attach the boundary $\partial E$ of the $j$-th diagram $H_R$ and the boundary $\partial I$ of the $j+1$-th one along their boundaries following the numbers $1, 2, 3, 4$, for each $j = 1, 2, \ldots, a_2 - 1$. For the disk $I$ of the first diagram $H_R$, attach the boundary $\partial E$ of the $a_1$-th diagram $H_L$. Continuing this process, we finally get a diagram by gluing $1+\sum_{i=1}^{n} a_i$ pieces of $H_L$, $H_R$ and $H_C$. We denote the resulting piece of a punctured Heegaard diagram by $H_{(p,q)}$.

We define $H_b$ ($b \in \mathbb{Z}$) to be another piece of a punctured Heegaard diagram constructed following the same argument using the word $LR^b\overline{L}$ when $b$ is non-negative and $L\overline{R}^{-b}\overline{L}$ otherwise.

Let $H_S$ and $H_T$ be the pieces of a punctured Heegaard diagram shown in Fig. 6 and 7, respectively.

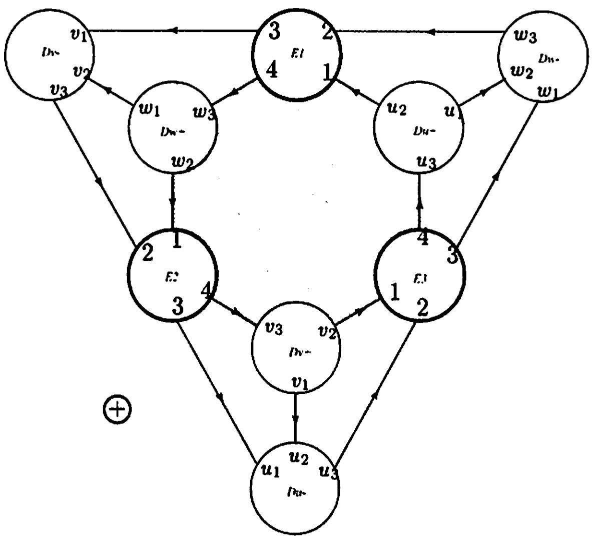


FIGURE 6. The piece $H_S$.

Let $g$ be a non-negative integer and $b$ be an integer. Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r)$ be pairs of mutually coprime integers such that $1 < \alpha_i$ and $0 < \beta_i < \alpha_i$ ($i = 1, 2, \ldots, r$).

Prepare $g+r-2$ copies $H_S^1, H_S^2, \ldots, H_S^{g+r-2}$ of the diagram $H_S$.

Attach the boundary $\partial E_3$ of the diagram $H_S^i$ to the boundary $\partial E_1$ of the diagram $H_S^{i+1}$ for $i = 1, 2, \ldots, g+r-2$. Then we get a piece of a punctured Heegaard diagram. Note that this diagram has $g+r+1$ boundary components. Attach the diagrams $H_{(p_j,q_j)}$ ($j = 1, 2, \ldots, r$), $H_b$ and the $g$ copies of the diagram $H_T$ to the boundary components of the resulting diagram, respectively. Now, we get a punctured Heegaard diagram and denote it by $H_{(g;b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_r,\beta_r))}$.

**Theorem 2.1.** *The punctured Heegaard diagram $H_{(g;b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_r,\beta_r))}$ corresponds to the Seifert fibered 3-manifold $S(g;b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_r,\beta_r))$ with a standard Spin$^c$ structure.*

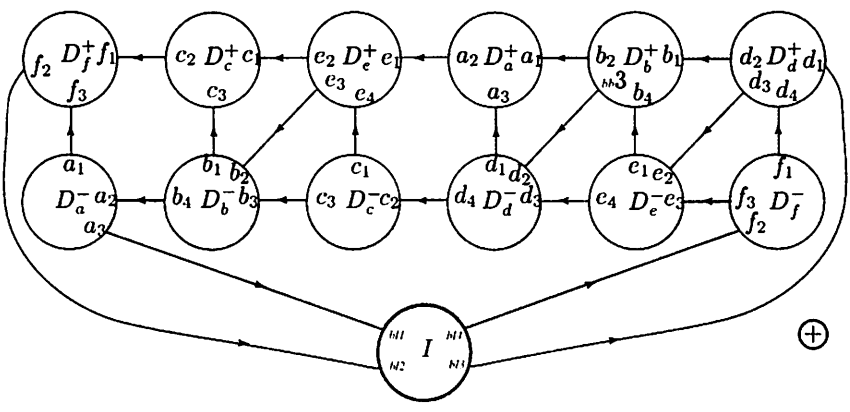

FIGURE 7. The piece $H_T$.

*Proof.* The idea of the proof is to construct the pieces of the punctured Heegaard diagram corresponding to the pieces of the DS-diagram constructed in [15] following the proof of Theorem 5.5.

Let $\pi$, $B$, $W$ and $e$ be as described above.

Set $A_i := \eta(e; \partial\eta(P))$. Recall that $e$ has the B part on one side and the W one on the other side. The key idea is to draw a simple closed curve $C$ in $A$ such that

(1) $C$ is isotopic to $e$ in $A$;
(2) $C \cap e \neq \emptyset$ and $C$ intersects $e$ transversely; and
(3) $C \cap \pi^{-1}(S(P)) \subset e \setminus \pi^{-1}(V(P))$.

Let $\mathcal{H}_R$ be a piece of DS-diagram (on the annulus) shown in Fig. 8 (i). (It is defined in [15].) The curve $e$ lies horizontally in the middle part of the diagram and its separates the diagram into B-part, on the upper side, and W-part, on the under side. Then the

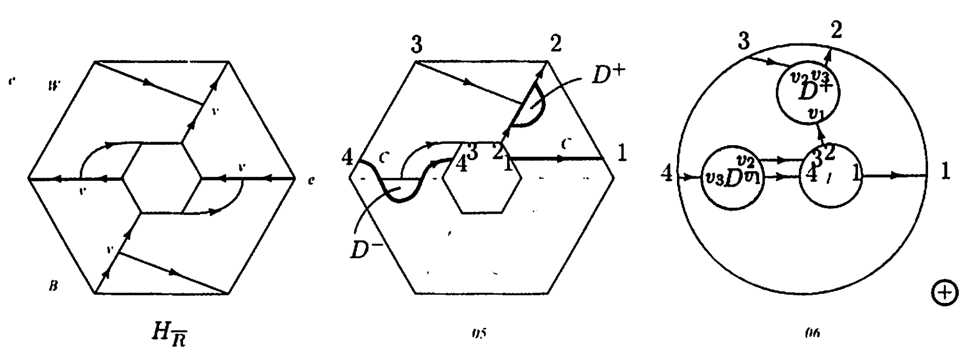


FIGURE 8. $H_R$.

intersection $C \cap \mathcal{H}_R$ is depicted by the bold lines in Fig. 8 (ii). The two curves $C \cap \mathcal{H}_R$ cut the annulus into two disks, the under piece of which corresponds to the joining disk.

Note that the disk $D_-$ shown in the figure is identified via the projection $\pi$ with $D_+$. Now we get a piece of a punctured Heegaard diagram. Fig. 9.

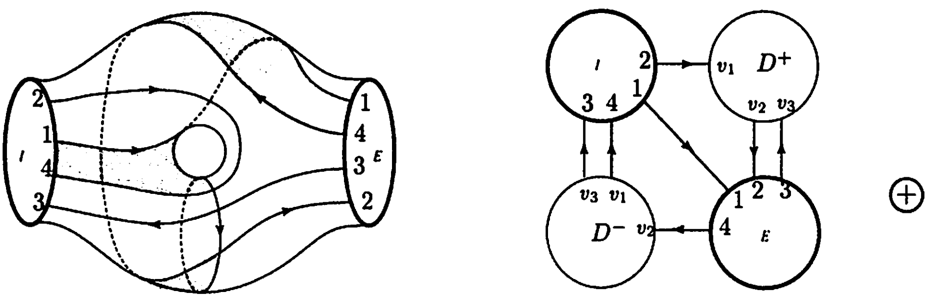


FIGURE 9. $H_R$.

For the other pieces shown in [15], we can apply the same argument. Cnsequently, we get the assertion. □

*Remark.* Once we forget the joining disk and regard the diagram $H_{(g;b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_g,\beta_g))}$, it gives a Heegaard diagram of the Seifert fibered manifold $S(g;b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_g,\beta_g))$. For each piece of the Heegaard diagram corresponding to the singular fiber, the diagram can be destabilized to be the diagram with genus one.

2.2. **Algorithm.** Let $(S_g;\alpha,\beta;D)$ be a punctured Heegaard diagram with $\partial M_{(S_g;\alpha,\beta;D)} \cong S^2$. Recall that once given a punctured Heegaard diagram, the Heegaard surface $S_g$ assumed to be naturally oriented as explained in Section 1. Let $F$ be a field and $\varphi : \mathbb{Z}[H_1(\widehat{M}_{(S;\alpha,\beta;D)})] \to F$ be a ring homomorphism. We can calculate the Reidemeister-Turaev torsion of the Spin$^c$ structure $[\widehat{\mathcal{V}}_{((S;\alpha,\beta;D))}] \in \mathrm{Spin}^c(\widehat{M}_{(S;\alpha,\beta;D)})$ carried by the flow-spine $P_{(S;\alpha,\beta;D)}$ in the folloiwng algorithmical way:

- **Step 1:** Orient $\alpha$ and $\beta$, and take base points of $\beta$ following the rule prescribed in Section 1.
- **Step 2:** Get a presentation $\langle x_1,\ldots,x_g \mid r_1,\ldots,r_g\rangle$ of $\pi_1(\widehat{M}_{(S;\alpha,\beta;D)},*)$ as in the rule in Section 1.
- **Step 3:** Find an arbitrary dual system $\gamma$ of $\beta$ in the diagram $(S;\alpha,\beta;D)$ and relate a word $y_i$ of $x_1,\ldots,x_g$ to each loop $\gamma_i$ in $\gamma$ in the same rule as in Step 2.
- **Step 4:** If there exist two integers $k,l \in \{1,\ldots,g\}$ such that all of $\det B_{k,l}$, $\varphi([y_l])-1$ and $\varphi([y_l])-1$ are nonzero, then we have

$$\begin{aligned}&\tau^{\varphi}\quad (S(g;b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_r,\beta_r)),\widehat{\mathcal{V}}_{st})\\ =&\ \pm\frac{\det B_{k,l}}{(\varphi([x_k])-1)(\varphi([y_l])-1)} \in F^{\times}/\pm 1,\end{aligned}$$

here $B_{k,l}$ is the $(k,l)$-minor of the matrix $\left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1\le i,j\le g}$. If there are not such integers $k$ and $l$, then it turns out that the twisted chain complex $C^{\varphi}(S(g;b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_r,\beta_r))$ is not acyclic, hence we have $\tau^{\varphi}=0$ by definition.

Remark that due to the results in Meng-Taubes [12] or Turaev [18], the above also gives an purely combinatorial algorithm to compute the Seiberg-Witten invariant of standard Spin$^c$ structure when the given Seifert fibered 3-manifold has the first homology group of infinite order.

## 3. On torus decomposition

### 3.1. Embedded torus appearing in Heegaard diagram.

**Lemma 3.1.** *Let $(S;\alpha,\beta)$ be a Heegaard diagram of a closed 3-manifold $M$. If there is a separationg simple closed curve $\gamma \subset S$ such that*

1. $\alpha \cap \gamma = \emptyset$;
2. *there are two integers $1 \leqslant i < j \leqslant g$ such that $\#(\beta_i \cap \gamma) = \#(\beta_j \cap \gamma) = 2$;*
3. *$\gamma$ is in general position with respect to $\beta_i \cup \beta_j$; and*
4. *if we set $\beta_i \cap \gamma = \{p_1, p_2\}$ and $\beta_j \cap \gamma = \{q_1, q_2\}$, they appear, without loss of generality, in the order as $p_1, q_1, p_2, q_2$ along the loop $\gamma$,*
5. *at the points $p_1$ and $p_2$ (resp. $q_1$ and $q_2$), $\gamma$ and $\beta_i$ (resp. $\beta_j$) intersects with different signs,*

*then there is a torus $T$ embedded in $M$ such that $T \cap S = \gamma$.*

*Proof.* Since the simple closed curve $\gamma$ does not intersect $\alpha$, $\gamma$ bounds a 2-disk $E_\alpha$ in the handlebody $H_\alpha$.

Consider the handlebody $H_\beta$. Since $\gamma$ intersects $\beta_i$ and $\beta_j$ twice respectively, $\gamma$ separates $\beta_i$ into two arcs $a_1$ and $a_2$, and $\beta_j$ into $b_1$ and $b_2$.

Let $a^+$ and $a^-$ be components of $\partial\eta(\beta_i; S) \setminus \gamma$ corresponding to the arc $a_1$. Let $b^+$ and $b^-$ be components of $\partial\eta(\beta_j; S) \setminus \gamma$ corresponding to the arc $b_1$.

Due to the conditions (4) and (5), the union

$$\gamma' = (\gamma \setminus (\eta(\beta_i; S) \cup \eta(\beta_j; S))) \cup a^+ \cup a^- \cup b^+ \cup b^-$$

is a simple closed curve on the sphere $\partial(\overline{H_\beta \setminus \sum_k D_{\beta_k}})$, and hence it bounds a disk $E_\beta$ in $\overline{H_\beta \setminus \sum_k D_{\beta_k}}$.

The disk $E_\beta$ can be regarded as a properly embedded disk in $H_\beta$. Now, identifying the boundary arcs $a^+$ with $a^-$ and $b^+$ with $b^-$ by naturally retracting the neighborhood $\eta(\beta_i; S)$ and $\eta(\beta_j; S)$ to $\beta_i$ and $\beta_j$, respectively, we get a once punctured torus $T_\beta$ such that $\partial T_\beta = \gamma$ by condition (5). Then the union $E_\alpha \cup T_\beta$ becomes a required embedded torus. □

3.2. **Closed normal o-graphs.** In this subsection, we describe the construction of closed normal o-graphs of Seifert fibered manifolds corresponding to the above method. For the definition and properties of o-graphs and inparticular closed normal o-graph See [2].

Let $\Gamma$ be a normal o-graph. Recall that $\Gamma$ is 4-regular, two opposite germs of edges incident to a vertex of $\Gamma$ are marked as being *over* the other two (which is marked as being *under*), as in link projections, and the directions of opposite edges match through the vertices.

Given a pair of mutually coprime positive integers $(\alpha, \beta)$, we define the piece of o-graph $\Gamma_{(\alpha,\beta)}$, called a *tail*, as follows.

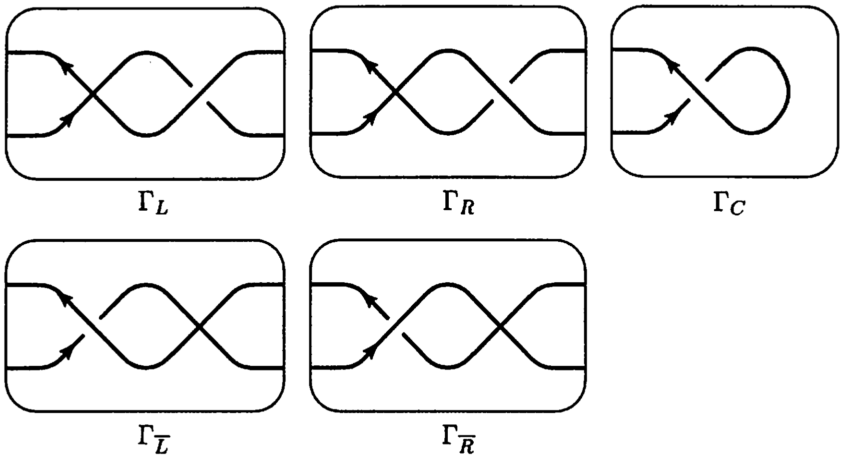


FIGURE 10. The pieces $\Gamma_L$, $\Gamma_R$, $\Gamma_{\overline{L}}$, $\Gamma_{\overline{R}}$ and $\Gamma_C$.

Given a word $A(p,q)$, where $q/p = [a_1, a_2, \ldots, a_n, 1]$, connect $a_1$ copies of the graphs $\Gamma_L$ horizontally. Next, take $a_2$ copies of the diagrams $\Gamma_R$ and connect them to the above pieces horizontally. Continuing this process, we finally cap off by $\Gamma_C$. Now get a piece of graph by gluing $1 + \sum_{i=1}^{n} a_i$ pieces of $\Gamma_L$, $\Gamma_R$ and $\Gamma_C$, and denote the resulting piece of an o-graph by $\Gamma_{(p,q)}$.

Now, Theorem 2.1 can be restated as follows.

**Theorem 3.2.** *An o-graph of any closed Seifert fibered manifold is constructed from tails and copies of the pieces $\Gamma_S$ and $\Gamma_T$ shown in Figures* 11 *and* 10. *More precisely, given a Seifert parameter $S(g; b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r))$, the required o-graph $\gamma_{S(g;b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_g,\beta_g))}$ is constructed from $g + r - 2$ copies of $\Gamma_S$, $g$ copies of $\Gamma_T$, $\Gamma_{(\alpha_1,\beta_1)}$, $\Gamma_{(\alpha_2,\beta_2)}$, $\ldots$, $\Gamma_{(\alpha_r,\beta_r)}$ and $\Gamma_b$.*

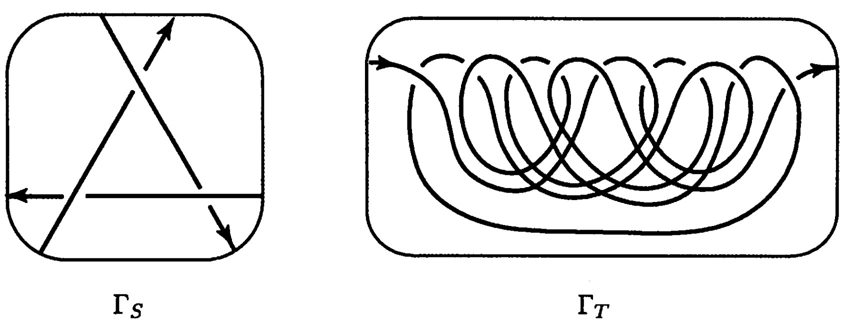


FIGURE 11. The pieces $\Gamma_S$ and $\Gamma_T$.

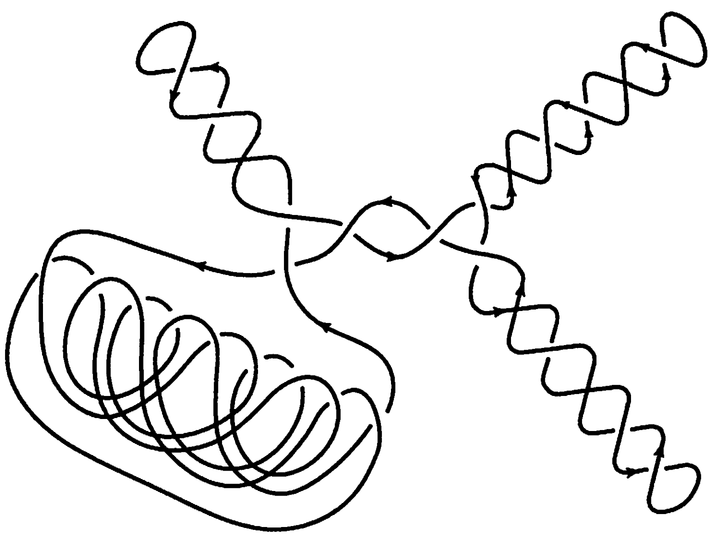

FIGURE 12. An o-graph of $S(1;0;(2,1),(3,1))$.

Recall that a graph is said to be 2-*connected* if is not connected after removing appropriate two edges. For a graph $G$, we denote by $V(G)$ the set of vertices of $G$, and by $E(G)$ the set of edges of $G$. Also, for a vertex $v$ of $G$, we denote by $\deg_G(v)$ the degree of $v$ in $G$, as usual.

**Lemma 3.3.** *Any connected 4-regular graph is 2-connected.*

*Proof.* Let $G$ be a connected but not 2-connected 4-regular graph. Then there is an edge $e$ such that $G \setminus e$ has two components $G_1$ and $G_2$, and we have

$$2 \cdot \#E(G_1) = \sum_{v \in V(G_1)} \deg_{G_1}(v) = 4 \cdot \#V(G_1) - 1.$$

This is impossible. □

By the proof of above lemma, we see that if a 4-regular graph is $k$-connected for an odd number $k \in \mathbb{N}$, then it is also $(k+1)$-connected.

*Definition.* We say that an o-graph is *prime* if it is not 4-connected.

As a direct corollary of the above Lemma 3.1 and Theorem 5.14 in [11], we have the following.

**Corollary 3.4.** *Let $\Gamma$ be an o-graph. If $\Gamma$ is not prime, then we have the following:*

(1) *there exists an embedded separating torus $T$ in $M$ and $M$ can be written as $M = N_1 \cup_T N_2$;*
(2) *the torus $T$ can be taken as a union of orbits of the flow carried by a flow-spine which has the o-graph $\Gamma$;*

(3) *For each sequence $W_i(L, R)$ of letters $L$ and $R$, the o-graph obtained by capping of the piese of $\Gamma_i$ corresponding to $N_i$ by the tail corresponding to the sequence represents a closed 3-manifold $M_i$ obtained by a Dehn filling on the torus boundary of $N_i$.*

The above corollary shows that the presentation of 3-manifolds via normal o-graph has a strong connection with the torus decomposition. Note that the first assertion is announced by Kouno in [Kou06] in the category of almost-special spines.

The construction of an o-graph of a Seifert fibered 3-manifold in Theorem 3.2 gives a good example of Corollary 3.4 (i). Recall that the pieces of an o-graph in the construction correspond to either (trice punctured sphere) $\times S^1$, (once punctured torus) $\times S^1$ or a fibered solid torus. When we connect two pieces, the connection is made by two edges which separate the resulting o-graph and this process corresponds to a gluing along boundary tori. This is a special case of Corollary 3.4.

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