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ON A SYMBOLIC ENCODING OF 3-MANIFOLDS

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ABSTRACT. We prove that an invariant of closed 3-manifolds defined by Endoh and Ishii, the *block number*, which measures the complexity of 3-manifolds via their flow-spines, equals the Heegaard genus, except in the case of S^3 and $S^2 \times S^1$. We also show that the underlying 3-manifold is uniquely determined by a neighborhood of the singularity of a flow-spine. This allows us to encode a closed 3-manifold by a sequence of signed labeled symbols. The behavior of the encoding under the connected sum and a criterion for reducibility are studied.

1. INTRODUCTION

Simple spines have been shown to be quite useful to study the combinatorial structure and invariants of 3-manifolds by many authors, see for example [1], [2], [3], [15], [16], [17] and [21]. Denote by $\mathcal{M}$ the set of all closed oriented 3-manifolds and by $\mathcal{CM}$ the set of all combed 3-manifolds, where a combed 3-manifold is a closed oriented 3-manifold endowed with a non-singular vector field. Using simple spines (especially *flow-spines* in the terminology of [12]), Endoh and Ishii introduced in [7] a non-negative function $Bl : \mathcal{M} \to \mathbb{Z}_{\geqslant 0}$ (resp. $Bl^* : \mathcal{CM} \to \mathbb{Z}_{\geqslant 0}$), called the *block number* of closed orientable 3-manifolds (resp. combed 3-manifolds). This invariant measures the complexity of one of the "simplest" non-singular vector fields on a 3-manifold (resp. a complexity of the combing). On the other hand, the more common invariant of 3-manifolds, the Heegaard genus $HG : \mathcal{M} \to \mathbb{Z}_{\geqslant 0}$, measures the complexity of one of the "simplest" Morse-Smale vector fields on the 3-manifold. In [7], the authors showed that the block numbers of all lens spaces, including S^3, are 1, with the exception of $S^1 \times S^2$, which has the block number 0, and that the block number is bounded below by the Heegaard genus, except for $S^1 \times S^2$. Moreover, they asked whether the block number equals the Heegaard genus, except in the case of S^3 and $S^1 \times S^2$. Endoh gave in [6] an affirmative answer to this problem in the case of 3-manifolds of Heegaard genus two. In this paper, we give a complete affirmative answer to this.

Theorem 1.1. *Every closed orientable 3-manifold M except $S^2 \times S^1$ and S^3 satisfies $Bl(M) = HG(M)$.*

It is interesting that the block number says that the simplest closed orientable 3-manifold is $S^2 \times S^1$, while the Heegaard genus says that it is S^3. In fact, for the Turaev-Viro invariant, which is defined as a certain state sum calculated on an arbitrary simple spine of a 3-manifold, normalization with respect to $S^2 \times S^1$ seems more natural than that with respect to S^3.

Via branched standard spines, we extend the concept of block number to all compact 3-manifolds with boundary. Let $\mathcal{N}$ be the set of compact 3-manifolds with boundary.

Then we define a non-negative function $Obl : \mathcal{N} \to \mathbb{Z}_{\geqslant 0}$, called *o-block number*. For the time being, we do not know whether this invariant is really new or this gives a new comprehension of other known invariants.

Each *transversely oriented* flow-spine of an oriented 3-manifold is encoded by a sequence of signed, labeled symbols such that every symbol in the sequence appears exactly twice, each symbol is labeled with l (light) or r (right) and so that each symbol appears once signed as $-$ (negative) and once signed as $+$ (positive), e.g. $v_1^{-l}v_2^{-r}v_3^{+r}v_1^{+l}v_4^{+r}v_5^{-l}v_4^{-r}v_5^{+l}v_3^{-r}v_2^{+r}$. This encoding is called an *E-datum*. If we restrict to the class of flow-spines which is standard, and E-datum determines a unique flow-spine, if it does exist, and hence a unique closed oriented 3-manifold, due to [2] and [4], but in general, a flow-spine may not be standard. In [2, Theorem 5.1.3], Benedetti and Petronio introduced an example of a pair of non-homeomorphic flow-spines having the same E-datum and left the problem in a following remark whether an E-datum determines a unique 3-manifold. We give a positive answer to this.

Theorem 1.2. *Let P_1 and P_2 be flow-spines of closed oriented 3-manifolds M_1 and M_2, respectively, such that they both have the same E-datum $\mathcal{E}$. Then M_1 is homeomorphic to M_2 via an orientation-preserving homeomorphism.*

This theorem implies that each E-datum encodes a unique closed oriented 3-manifold and hence E-datum turn out to work particularly well for the algorithmic and purely symbolic study of closed oriented 3-manifolds and their invariants.

In the last section, we give an E-datum of the connected sum of two closed 3-manifolds. We also give a criterion for the 3-manifold represented by an E-datum to be reducible.

Theorem 1.3. (1) *Let $\mathcal{E}_i$ be an E-datum of a closed oriented 3-manifold M_i for $i = 1, 2$. Then an E-datum of the connected sum $M_1 \# M_2$ is obtained by*
$p_1^{-l}p_2^{-r}\mathcal{E}_1 p_2^{+r}q_1^{+l}q_1^{-l}q_2^{-r}\mathcal{E}_2 q_2^{+r}p_1^{+l}$ *(and by* $p_1^{+r}p_2^{+l}\mathcal{E}_1 p_2^{-l}q_1^{-r}q_1^{+r}q_2^{+l}\mathcal{E}_2 q_2^{-l}p_1^{-r}$*),*
where p_i, q_i $(i = 1, 2)$ are symbols which appear in neither $\mathcal{E}_1$ nor $\mathcal{E}_2$.

(2) *Let M be a closed oriented 3-manifold M which does not contain an embedded non-separating 2-sphere. Let $\mathcal{E}$ be an E-datum of M of the following form:*

$$p_1^{-l}p_2^{-r}\mathcal{E}_1 p_2^{+r}q_1^{+l}q_1^{-l}q_2^{-r}\mathcal{E}_2 q_2^{+r}p_1^{+l} (\text{or } p_1^{+r}p_2^{+l}\mathcal{E}_1 p_2^{-l}q_1^{-r}q_1^{+r}q_2^{+l}\mathcal{E}_2 q_2^{-l}p_1^{-r}),$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are subwords of $\mathcal{E}$. Then no symbol in $\mathcal{E}_1$ appears in $\mathcal{E}_2$ and vice versa, and there exist closed oriented 3-manifolds M_1 and M_2 encoded by the E-data $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Moreover, these satisfy $M = M_1 \# M_2$.

We stress that *standard* flow-spines correspond bijectively to *closed normal o-graphs*, see [2], and we can state the above theorem in this terminology, see the remark following the proof of Theorem 1.3. A formula for (possibly non-closed) *o-graphs* corresponding to Theorem 1.3.1, can be found in [19]. The operation in [19], though, does not preserve the condition for a normal o-graph to be closed.

2. Preliminaries

Throughout this paper, we will denote the interior of X by $\mathrm{Int}X$, the closure of X by $\overline{X}$ and the number of components of X by $\#X$. We will use $N(X)$ to denote a regular neighborhood of X. By 3-manifold, we always mean a *connected, compact* and *oriented* one, with or without boundary.

2.1. **branched spines and flow-spines.** Let N be a compact oriented 3-manifold. A branched surface $P \subset N$ is a union of finitely many compact smooth surfaces glued together to form a compact subspace locally modeled on one of the three possibilities in Fig. 1. Note that the general definition of branched surface allows more sheets than just

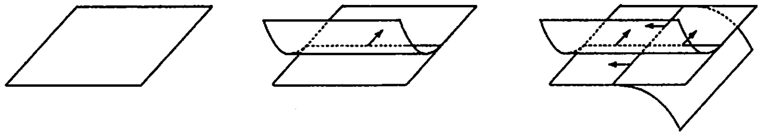


FIGURE 1. Local pictures of a branched surface

two on one side and one on the other side, and we only considers this situation (which is general and stable, i.e. corresponds to an open dense set in the space of branched surfaces).

The *branched locus* $S(P)$ of P is the set of points none of whose neighborhoods (in P) is a disk. $S(P)$ is a collection of smooth immersed curve in P. Let $V(P)$ be a union of double points of $S(P)$. We associate with every component of $S(P) \setminus V(P)$ a normal vector (in P) pointing in the direction of the cusp, as shown in Fig. 1. We call a component of $P \setminus S(P)$ a *sector* of P. Let R be a sector of P. If all branch directions along $\partial\overline{R}$ point out from R, $P \setminus R$ is still a branched surface, see Fig. 2 (i). One can

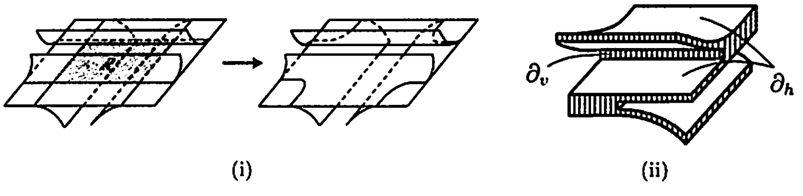


FIGURE 2. (i) Removable sector (ii) A regular neighborhood of a branched surface

regard $N(P)$ as an interval bundle over P as drawn in Fig. 2 (ii). The boundary $\partial N(P)$ decomposes into two parts: the endpoints of the fibers, $\partial_h N(P)$, and the rest, $\partial_v N(P)$. In this paper, all branched surfaces are assumed to be *transversely oriented*, that is, P is equipped with a global orientation on the 1-foliation of $N(P)$ whose leaves are fibers of $N(B)$. Refer to [8] and [18] for more details about branched surfaces.

A branched surface $P \subset N$ is called a *branched spine* (of N) if N collapses onto P. A branched spine P is naturally stratified as $V(P) \subset S(P) \subset P$. A branched spine P is called *standard* if this stratification induces a CW decomposition of P, namely, there is no loop and sectors are disks. See [2] for a precise definition. If P is a branched spine of a compact 3-manifold N with $\partial N = S^2$, then P is also called a branched spine of the closed 3-manifold M obtained from N by attaching a 3-ball to the unique 2-sphere boundary. A branched spine of a closed 3-manifold is called a *flow-spine* if $\partial_v N(P)$ is an annulus.

In [2], Benedetti and Petronio proved that every orientable 3-manifold admits a branched (standard) spine and it naturally encodes a well-defined homotopy class of vector fields, which is called the *concave traversing field*, on the ambient manifold. We require the flow intersects P at $x \in P$ in the same direction as the fixed transverse orientation. In the case where P is a flow-spine of a closed oriented 3-manifold M, one can extend the concave traversing field, whose orbits are the I-fibers of the regular neighborhood of the spine, to the whole of M.

Each vertex $v \in V(P)$ is classified into the two types l and r as shown in Fig. 3. In the figure, $\mathbf{w}_x \in T_xM$ and $(\mathbf{u}_x, \mathbf{v}_x, \mathbf{w}_x) \in T_xP \oplus T_xP \oplus T_xM$ are compatible with the transverse orientation of P and the orientation of M, respectively. A *code* is a map $\gamma_P : V(P) \to \{l, r\}$ such that $\gamma_P(v) = l$ (resp. r) if v is a vertex of l-type (resp. r-type).

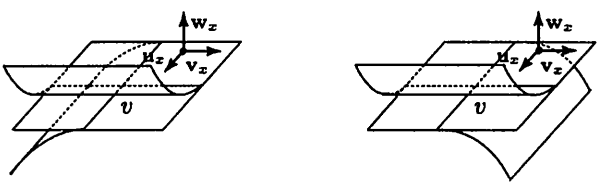


FIGURE 3. The two types of vertices

An *oriented combed 3-manifold* is a pair $(M, \mathcal{V})$ of an oriented closed 3-manifold M and a non-singular vector field $\mathcal{V}$ on it. Two combed 3-manifolds $(M_1, \mathcal{V}_1)$ and $(M_2, \mathcal{V}_2)$ are said to be *equivalent* if there is an orientation preserving diffeomorphism $h : M_1 \to M_2$ such that $h_*(\mathcal{V}_1)$ is homotopic to $\mathcal{V}_2$ within the class of non-singular vector fields on M_2. In [12], Ishii showed that every non-singular vector field on a closed 3-manifold M is carried by a flow-spine in M.

2.2. **DS-diagrams and punctured Heegaard diagrams.** We recall the two types of diagrams, *DS-diagrams with E-cycle* and *punctured Heegaard diagrams*, of closed 3-manifolds (and its combing), and an encoding of 3-manifolds (and its combing) by sequences of symbols derived from these diagrams, *E-datum*.

First, we review the notions and the known results on of DS-diagrams and E-data. See [11] [2] and [7] for more details and examples on the topic.

We call a triple $\Delta = (G, \phi, P)$ a *DS-diagram* if it satisfies the following: (i) $G \subset \partial B^3$ is a trivalent graph; (ii) P is thickenable and the boundary of the thickening is S^2; (iii) $\phi : \partial B^3 \to P$ is a local homeomorphism such that $\phi^{-1}(S(P)) = G$ and $\phi^{-1}(V(P)) = V(G)$, where $V(G)$ is the set of the vertices of the graph G; and (iv) For $x \in P$, $\phi^{-1}(x)$ consists of 2, 3 or 4 points according as $x \in P \setminus S(P)$, $x \in S(P) \setminus V(P)$ or $x \in V(P)$.

Given a DS-diagram Δ, the quotient space $M(\Delta) = B/\phi$ becomes a closed (possibly non-orientable) 3-manifold. A DS-diagram can be regarded as a presentation of 3-manifolds obtained by cutting them along simple spines.

Let $\Delta = (G, \phi, P)$ be a DS-diagram. Let e be a cycle of the graph G and split the sphere on which G lies as $S^2 \setminus e = \Sigma_+ \cup \Sigma_-$ (two open disks). Then e is called an *E-cycle* if it satisfies the following: (i) $\#(\Sigma_\pm \cap \phi^{-1}(x)) = 1$ for each point x on $P \setminus S(P)$; and (ii) $\#(e \cap \phi^{-1}(x)) = \#(\Sigma_\pm \cap \phi^{-1}(x)) = 1$ for each point x on $S(P) \setminus V(P)$.

A DS-diagram obtained by cutting a 3-manifold M along a flow-spine P naturally has an E-cycle, which consists of the cusp points of $M \setminus P$. Conversely, for DS-diagram with E-cycle $\Delta(G, \phi, P; e)$, the spine P naturally has the structure of a flow-spine.

For DS-diagram with E-cycle $\Delta(G, \phi, P; e)$, the spine P naturally has the structure of a flow-spine. In this paper, we assume that every DS-diagram has an E-cycle. When we consider a DS-diagram $\Delta(G, \phi, P; e)$ of an orientable 3-manifold M, we always assume that e and S^2 on which the graph G is drawn are oriented and the restriction of the orientation of S^2 to the component Σ_+ of $S^2 \setminus e$ satisfies that $e = \partial\Sigma_+$ in the oriented sense. Then we call Σ_+ (resp. Σ_-) the *positive (resp. negative) region.* Moreover, B^3 is assumed to be given an orientation compatible with the orientation of $\partial B^3 = S^2$. In this way, a DS-diagram with E-cycle defines an oriented 3-manifold B^3/ϕ. The transverse orientation of P is also naturally defined by the DS-diagram, see [12] and [7].

For each vertex v of the spine $P \subset M(\Delta)$, there are exactly two vertices v^+ and v^- on the E-cycle e of the graph G such that $\phi(v^+) = \phi(v^-) = v$. These two vertices are characterized by the condition that $v_\pm \in \Sigma_\pm$.

Suppose $V(P)$ consists of n points $v_1, v_2, \dots, v_n$. Recall that each vertex of a branched surface is classified into two types l and r. An *E-datum* $\mathcal{E}(\Delta) = (\gamma_P, \mathcal{A}(\Delta))$ of $\Delta = (G, \phi, P; e)$ is a pair of the code $\gamma_P : V(P) \to \{l, r\}$ and the arrangement $\mathcal{A}(\Delta)$ of $2n$ vertices $v_k^\pm$ on the oriented circle e, e.g. $(v_1^- v_2^- v_3^+ v_1^+ v_4^+ v_5^- v_4^- v_5^+ v_3^- v_2^+$, $\gamma(v_1) = l, \gamma(v_2) = r, \gamma(v_3) = r, \gamma(v_4) = r, \gamma(v_5) = l)$. We usually denote v_i^l and v_i^r to express that the code of v_i is l and r, respectively. Then we can write an E-datum as a signed, labeled sequence of symbols as explained in the introduction.

Consider the virtual knot-like descriptions of flow spines, called *closed normal o-graphs*, introduced in [13], [1] and [2], see Fig. 4. Then the E-data (resp. the arrangements)

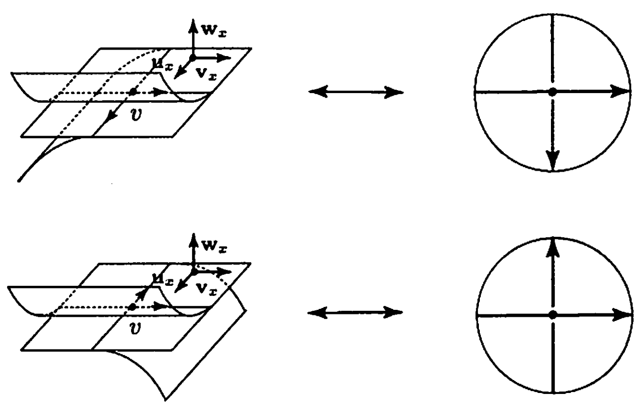


FIGURE 4. The local pictures of a branched spine and an o-graph

of the flow-spines are regarded as the *signed Gauss codes* (resp. *Gauss codes*) of the corresponding closed normal o-graphs, refer to [14]. Fig. 5 shows the virtual knot-like description of an E-datum.

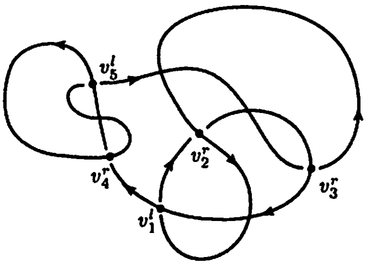


FIGURE 5. The o-graph with E-datum $v_1^{-l}v_2^{-r}v_3^{+r}v_1^{+l}v_4^{+r}v_5^{-l}v_4^{-r}v_5^{+l}v_3^{-r}v_2^{+r}$.

A *virtual E-datum* is a sequence of signed, labeled symbols such that every symbol in the sequence appears exactly twice, each symbol is labeled with l or r and so that each symbol appears once signed as negative and once signed as positive. Note that not every virtual E-datum encodes a *closed* oriented 3-manifold.

We say that an E-datum is *negatively (resp. positively) normalized* if it starts with a negative (resp. positive) symbol and ends with a positive (resp. negative) symbol.

Definition. An *o-datum* is a union of sequences of signed, labeled symbols such that every symbol appears in the union of sequences exactly twice, each symbol is labeled with l or r and so that each symbol appears once signed as negative and once signed as positive. Moreover, we require that if we divide the sequences into two unions, there exists a symbol which appears once in one union and once in the other union.

Since each o-datum gives rise to a unique (connected) normal o-graph and every normal o-graph determines a unique 3-manifolds with boundary, due to [2], an o-datum encodes a unique 3-manifold with boundary. Notice that the number of sequences of an o-datum is equal to that of components of the corresponding o-graph regarded as a virtual knot.

In the contrary direction, the following important fact holds:

Theorem 2.1 ([2]). *Every oriented 3-manifold with boundary admits a branched standard spine.*

Each branched standard spine is naturally encoded by a *normal o-graph*, which is defined in the same way as a closed normal o-graph described above. Hence we can encode any oriented 3-manifolds with boundary by o-data.

Given an E-datum $\mathcal{E}$. A consecutive (in the sense of cyclical order) sequence of symbols in $\mathcal{E}$ is called a *subword* of it. A maximal subword in which all symbols are positively (resp. negatively) signed is called a *positive block* (resp. *negative block*) of $\mathcal{E}$. The *block number* $bl(\mathcal{E})$ of the E-datum $\mathcal{E}$ is defined to be the number of positive blocks included in $\mathcal{E}$ if $\mathcal{E}$ is not empty. In the case where $\mathcal{E}$ is empty, we set $bl(\mathcal{E}) = 0$. Note that a spine giving rise to empty E-datum cannnot be standard.

Definition. The *block number* $Bl(M)$ (resp. $Bl^*(M, \mathcal{V})$) of an orientable closed 3-manifold M (resp. a combed 3-manifold $(M, \mathcal{V})$) is the minimum of the block numbers of all E-data whose underlying 3-manifold is M (resp. $(M, \mathcal{V})$).

When we use, again, the virtual knot-like description of flow spines, the block number can be regarded as the *bridge number* of the diagram, e.g. the block number of the E-datum shown in Fig. 5 is three.

Theorem 2.2. (1) ([7]) *$Bl(M) = 0$ if and only if M is homeomorphic to $S^2 \times S^1$. $Bl(M) = 1$ if and only if M is homeomorphic to S^3 or a lens space.*

(2) ([7]) *$HG(M) \leqslant Bl(M)$ for every closed orientable 3-manifold M except for $S^2 \times S^1$.*

(3) ([6]) *$HG(M) = Bl(M)$ for any closed orientable 3-manifold of Heegaard genus two.*

We can define in an analogous way an invariant for oriented 3-manifolds with boundary taking the bridge number of its normal o-graphs. The *o-block number* of an o-data $\mathcal{E}$ is the sum of the number of positive blocks included in each sequence in $\mathcal{E}$.

Definition. The *o-block number* $Obl(N)$ of a compact 3-manifold N is the minimum of the o-block numbers of all o-data whose underlying 3-manifold is N.

In a word, o-block number of a compact 3-manifold with boundary measures the complexity of one of the "simplest" concave vector fields on the manifold.

Next, we recall the definition of *punctured Heegaard diagram* defined in [16] in more general settings.

By a *Heegaard diagram* we mean a triple $(S; \alpha, \beta)$ where S is a closed, connected, orientable surface and $\alpha := \bigcup_{i=1}^{g} \alpha_i$ and $\beta := \bigcup_{i=1}^{g} \beta_i$ are compact, mutually transverse 1-manifolds on S such that both $S \setminus \alpha$ and $S \setminus \beta$ are a $2g$-times punctured sphere. A Heegaard diagram gives rise to a closed 3-manifold $M = M_{(S;\alpha,\beta)}$ obtained by attaching 2-handles $H_{\alpha_1}, \dots, H_{\alpha_g}$ and $H_{\beta_1}, \dots, H_{\beta_g}$ to $S \times [-1, 1]$ along the curves $\alpha_1 \times \{-1\}, \dots, \alpha_g \times \{-1\}$ and $\beta_1 \times \{1\}, \dots, \beta_g \times \{1\}$, respectively, and then attaching two 3-handles along the resulting 2-sphere boundary components. The decomposition of M by $S \times \{0\}$ is the associated Heegaard splitting of M and the genus of S is called the *genus* of the splitting. We will denote the core disk of H_{α_i} (resp. H_{β_j}) (fairly extended so that its boundary is on S) by D_{α_i} (resp. D_{β_j}). An *oriented* Heegaard diagram is an oriented triad $(S; \alpha, \beta)$ with fixed positive normal to S points toward the β-side.

Given an oriented Heegaard diagram $(S; \alpha := \bigcup_{i=1}^{g} \alpha_i, \beta := \bigcup_{j=1}^{g} \beta_j)$. Let D be a component of $S \setminus (\alpha \cup \beta)$. Then D is said to be *joining* if it satisfies the following:

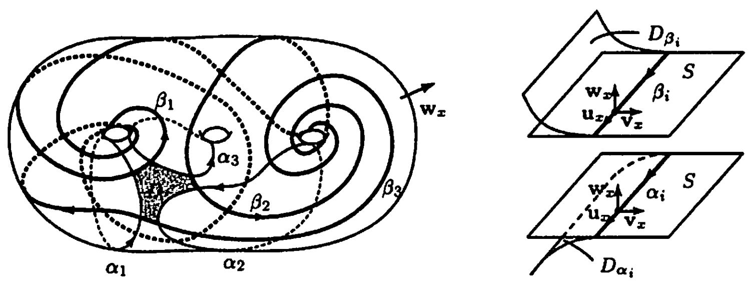
(1) $\partial \overline{D}$ is a simple closed curve, where the closure is taken in the surface S;

(2). $\partial \overline{D} \cap \alpha_j$ (resp. $\partial \overline{D} \cap \beta_j$) is an connected arc; and

(3) for each $x \in \partial \overline{D} \cap \alpha_i$ (resp. $x \in \partial \overline{D} \cap \beta_j$), let $u_x \in T_x\alpha_i$ (resp. $u_x \in T_x\beta_j$) be compatible with the orientation of α_i (resp. β_j) and $v_x \in T_xS \setminus T_x\alpha_i$ (resp. $v_x \in T_xS \setminus T_x\beta_j$) point out of D, then (u_x, v_x) is compatible with the fixed orientation of S.

We call an oriented Heegaard diagram $(S; \alpha, \beta)$ with joining disk D a *punctured Heegaard diagram* and denote it by $(S; \alpha, \beta; D)$.

Fig. 6 (i) illustrates an example of a punctured Heegaard diagram. When we depict a punctured Heegaard diagram, we always assume that the positive normal to S points outward from the depicted surface S. In the diagram the joining disk D is colored gray.



(i) (ii)

FIGURE 6. (i) A punctured Heegaard diagram (ii) Branching structure along α_i and β_i

For each oriented Heegaard diagram $(S; \alpha, \beta)$ of a 3-manifold M, we can associate an oriented branch structure with $B := S \cup (\bigcup_i D_{\alpha_i}) \cup (\bigcup_j D_{\beta_j})$ in such a way that if $u_x \in T_x\alpha_i$ (resp. $u_x \in T_x\beta_i$) is compatible with the orientation of the slope α_i (resp. β_i) and $v_x \in T_xS$ is such a tangent vector that (u_x, v_x) is compatible with the fixed orientation of S for $x \in \alpha$ ($x \in \beta$), then v_x is the branch direction. See Fig. 6 (ii).

Assume that there is a joining disk D of $(S; \alpha, \beta)$. Then D is a removable face of a branched surface B and by construction, $P = B \setminus D$ is a flow-spine of M equipped with a transverse orientation induced from that of S. In fact, the branched locus of P is the one obtained by smoothing $\overline{(\alpha \cup \beta) \setminus \partial\overline{D}}$ around the corners $\alpha \cap \beta \cap \partial\overline{D}$, and we can follow the branched locus using the slopes $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ up to reordering. Hence we can read the E-datum of the flow-spine P, and it is easy to check that following along α_i (resp. β_i) gives negative (resp. positive) vertices for the E-datum. Therefore the block number of the E-data is less than or equal to the genus of S. This fact plays an important role in the proof of Theorems 1.1 and 1.2. The E-datum corresponding to the diagram in Fig. 6 is $v_1^{-l}v_2^{-r}v_3^{+r}v_1^{+l}v_4^{+r}v_5^{-l}v_4^{-r}v_5^{+l}v_3^{-r}v_2^{+r}$. Given a DS-diagram with E-cycle of block number $g > 0$, there is a manner to construct a Heegaard splitting of genus g of the ambient 3-manifold, see [15].

Theorem 2.3 ([16]). *Any flow-spine of block number g having at least one vertex can be constructed from a punctured Heegaard diagram of genus g in the above way.*

Reall that flow-spines need not be standard. Given a flow-spine of block number $g > 0$, the idea of the proof is to draw a loop $C \subset P$ in the neighborhood $N(S(P)) \subset P$ of the branched locus $S(P)$ such that i) $S(P) \subset N(C) \subset P$; ii) $C \cap \partial N(S(P)) = \emptyset$; and C intersects $S(P)$ g times transversely at components of $S(P) \setminus V(P)$, see the left-hand side of Fig. 8. Since P is a flow-spine, such a simple closed curve must exist. Now, there is a closed disk D in M such that $\mathrm{Int}D \cap P = \emptyset$ and $\partial D = C$, see the right-hand side of Fig. 8, and the union $B = P \cap D$ can be regarded as a Heegaard surface S of genus g with complete systems of meridian disks $D_1, \ldots, D_g$ and $D'_1, \ldots, D'_g$ of the handlebodies into which S splits, and D is the joining disk of the Heegaard splitting.

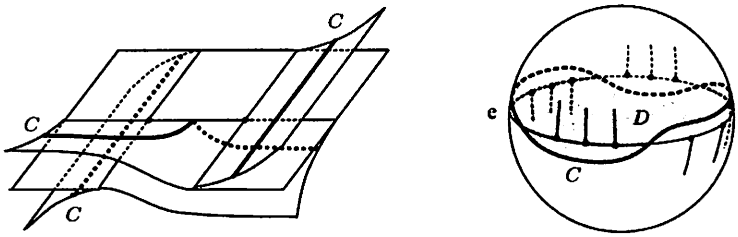


FIGURE 7. From a flow-spine to a Heegaard surface with meridian systems

3. THE HEEGAARD GENUS AND THE BLOCK NUMBER

Proof of Theorem 1.1. By Theorem 2.2.2, we only need to prove that $HG(M) \leqslant Bl(M)$ except in the case of S^3.

Let M be an orientable closed 3-manifold of Heegaard genus g such that $M \not\cong S^3$, hence $g > 0$. Let $(S; \alpha := \bigcup_i \alpha_i, \beta := \bigcup_j \beta_j)$ be a genus g Heegaard diagram of M. Take a small closed disk D in $S \setminus (N(\alpha) \cup N(\beta))$. We identify D with $\{z \in \mathbb{C} \mid |z| \leqslant 1\}$. Take an arbitrary point p_i (resp. q_j) on each component α_i (resp. β_j). Let $p'_i := e^{\frac{2i\pi\sqrt{-1}}{g}} \in \partial D$ and $q'_j := e^{\frac{(2j+1)\pi\sqrt{-1}}{g}} \in \partial D$ for $1 \leqslant i, j \leqslant g$. Let $\{\lambda_i\}_{i=1}^g$ (resp. $\{\mu_j\}_{i=1}^g$) be a collection of simple arcs on $S \setminus \mathrm{Int}D$ such that

(1) λ_i connects p_i to p'_i (resp. μ_j connects q_j to q'_j) for $1 \leqslant i \leqslant g$ (resp. $1 \leqslant j \leqslant g$),
(2) $\alpha \cap \mathrm{Int}\lambda_i = \emptyset$ (resp. $\beta \cap \mathrm{Int}\mu_j = \emptyset$) for $1 \leqslant i \leqslant g$ (resp. $1 \leqslant j \leqslant g$),
(3) $D \cap \mathrm{Int}\lambda_i = \emptyset$ (resp. $D \cap \mathrm{Int}\mu_j = \emptyset$) for $1 \leqslant i \leqslant g$ (resp. $1 \leqslant j \leqslant g$), and
(4) $\lambda_{i_1} \cap \lambda_{i_2} = \emptyset$ (resp. $\mu_{j_1} \cap \mu_{j_2} = \emptyset$) for $1 \leqslant i_1, i_2 \leqslant g$ (resp. $1 \leqslant j_1, j_2 \leqslant g$).

Then $\partial N(\lambda_i) \setminus \alpha_i$ consists of two open arcs c_i and c'_i, and we can assume that $c_i \cap D \neq \emptyset$. Set $\alpha'_i := (\alpha_i \setminus \mathrm{Int}N(\lambda_i)) \cup c_i$ for $1 \leqslant i \leqslant g$ and $\alpha' := \bigcup_i \alpha'_i$. Deform β to $\beta' := \bigcup_j \beta'_j$ in the same way using $\{\mu_i\}_{i=1}^g$. Since $(S; \alpha', \beta')$ is obtained from $(S; \alpha, \beta)$ by isotopy moves for the Heegaard diagram the underlying 3-manifold does not change. Finally, deform the diagram $(S; \alpha', \beta')$ in $N(\partial D)$ as shown in Fig. 8. Again, the underlying

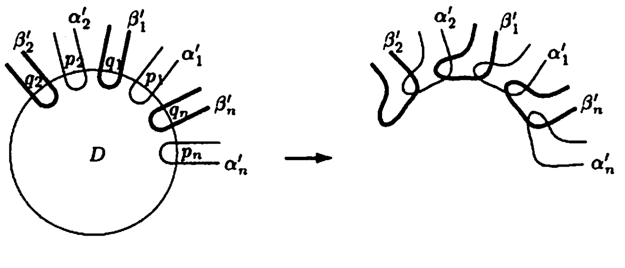


FIGURE 8. The deformation of α' and β'

3-manifold does not vary under this deformation of the diagram and the disk component of $S \setminus (\alpha' \cup \beta')$ contained in D becomes a joining disk after one equips an orientation of the resulting Heegaard diagram in an appropriate way. Let $\mathcal{E}$ be the E-datum of the flow-spine obtained from the above punctured Heegaard diagram. As we explained in the previous section, we have $bl(\mathcal{E}) \leqslant g$ and hence $Bl(M) \leqslant HG(M)$. This completes the proof.

The above proof provides the following corollary.

Corollary 3.1. *Every closed oriented 3-manifold admits an E-datum of the form*

$$v_1^{-l}\mathcal{V}_1 w_1^{-r} w_1^{+r}\mathcal{W}_1 v_1^{+l} v_2^{-l}\mathcal{V}_2 w_2^{-r} w_2^{+r}\mathcal{W}_2 v_2^{+l} \cdots v_g^{-l}\mathcal{V}_g w_g^{-r} w_g^{+r}\mathcal{W}_g v_g^{+l},$$

where $\mathcal{V}_i$ is a negative subword and $\mathcal{W}_i$ is a positive subword for $i = 1, \ldots, g$.

Proof. This is straightforward from the definition of a code and the above construction of a punctured Heegaard diagram. □

Question 3.2. *Does the o-block number of a compact oriented 3-manifold with non-empty boundary also have a relation to its Heegaard genus?*

4. An E-data determines a unique underlying 3-manifold

In [2, Theorem 5.1.3], Benedetti and Petronio proved that there exists pairs of non-homoemorphic flow-spines having the same E-datum $v_1^{-l}v_2^{-r}v_2^{+r}v_1^{+l}$. In this section, we prove that the underlying 3-manifold is uniquely determined by E-datum no matter though flow-spines having the E-datum may not be unique. We start with two lemmas on Heegaard diagrams.

Lemma 4.1. *Let $(S; \alpha := \bigcup_{j=1}^{g}, \beta := \bigcup_{j=1}^{g})$ be a Heegaard diagram of a closed orientable 3-manifold M. If there exists a non-separating simple closed curve $C \subset S$ such that $C \cap (\alpha \cup \beta) = \emptyset$, then $M = M'\#(S^2 \times S^1)$ for some closed orientable 3-manifold M' (possibly $M' \cong S^3$), and $(S'; \bigcup_{j\neq i_0} \alpha_j, \bigcup_{j\neq j_0} \beta_j)$ is a Heegaard diagram of M', where S' is obtained from S by compressing along C and $i_0, j_0 \in \{1, \ldots, g\}$ are some integers.*

Proof. Let D_1 and D_2 be disks spanned by C and properly embedded in the handlebodies H_1 and H_2 into which S splits M in negative and positive sides of S. Since C is non-separating in S, the sphere $\Sigma := D_1 \cup D_2$ does not separate M. Therefore M can be decomposed as $M = M'\#(S^2 \times S^1)$, where M' might be S^3, see [10].

Let ℓ be a simple closed curve properly embedded in M such that ℓ meets Σ transversely in a single point. M' is obtained by attaching a 3-ball along the resulting 2-sphere boundary of $\overline{M \setminus N(\Sigma \cap \ell)}$. It follows that M' is obtained by adding two 3-balls along the two 2-sphere boundary components of $\overline{M \setminus N(\Sigma)}$. Set $H_1' = \overline{H_1 \setminus N(D_1)}$ and $H_2' = \overline{H_2 \setminus N(D_2)}$. Let $\varphi : \partial H_1 \to \partial H_2$ be a homeomorphism corresponding to the given Heegaard diagram. Let D_1^+, D_1^- (resp. D_2^+, D_2^-) be the two frontiers of $N(D_1)$ (resp. $N(D_2)$) in H_1 (resp. H_2) such that $\varphi(\partial D_1^{\pm}) = \partial D_2^{\pm}$. Let $\overline{\varphi} := \varphi|_{\partial H_1 \cap \partial H_1'}$. Then $(H_1' \cup H_2')/\overline{\varphi} = \overline{M \setminus N(\Sigma)}$. Since the homeomorphism between the boundaries of closed disks extends to a unique homeomorphism between the disks up to isotopy, the map $\overline{\varphi}$ uniquely extends to a homeomorphism $\varphi' : \partial H_1' \to \partial H_2'$. We claim that $M' \cong (H_1' \cup H_2')/\varphi'$. In fact, $M' \cong (H_1' \cup H_2')/\varphi'$ is obtained from $(H_1' \cup H_2')/\overline{\varphi}$ by identifying D_1^+ with D_2^+ and D_2^- with D_2^-, this produces the same manifold as the one obtained by adding 3-balls to $D_1^+ \cup D_2^+$ and $D_1^- \cap D_2^-$. Now $S' := \partial H_1'$ is obtained from

S by compressing along C, hence we obtain the desired Heegaard diagram removing redundant meridian disks $D_{\alpha_{i_0}}$ and $D_{\beta_{j_0}}$. □

Let F be a compact planar (possibly disconnected) surface such that every component has non-empty boundary. Set $n(F) := \#\partial F - \#F$. A system $C_1, \dots, C_{n(F)}$ of mutually disjoint simple closed curves embedded in Int$F$ is said to be *boundary separating* if $\#(F') \cap \partial F = 1$ for each component F' of $F \setminus C_k$.

Lemma 4.2. *Let $(S; \alpha := \bigcup_{j=1}^{g}, \beta := \bigcup_{j=1}^{g})$ be a Heegaard diagram of a closed orientable 3-manifold M such that $\alpha \cup \beta$ is connected. Let F be the union of all non-disk components $F_1, \dots, F_r$ of the closure of $S \setminus (\alpha \cup \beta)$ in the path metric. Let $C_1, \dots, C_{n(F)}$ be a boundary separating system of F. Then $M = M' \#(\#_{n(F)}(S^2 \times S^1))$ for some closed orientable 3-manifold M' (possibly $M' \cong S^3$), and M' has a Heegaard diagram $(S'; \bigcup_{j \in I} \alpha_j, \bigcup_{j \in J} \beta_j)$, where S' is obtained from S by compressing along $C_1, \dots, C_{n(F)}$ and I (resp. J) is a subset of $\{1, \dots, g\}$ with order $g - n(F)$.*

Proof. We claim that each loop C_i is non-separating in S. Suppose that C_i were separating. since $\alpha \cup \beta$ is connected in S, then $S \setminus C_i$ separates a disk from S and the disk does not intersect $\alpha \cup \beta$, hence contained in the component of $\overline{S \setminus (\alpha \cup \beta)}$ in which C_i lies. This is a contradiction. Now the assertion follows by successively applying Lemma 4.2. □

Proof of Theorem 1.2. Let P_1 and P_2 be non-homeomorphic flow-spines giving the same E-datum $\mathcal{E}$. Due to Theorem 5.1, we may assume that the block number g of $\mathcal{E}$ is greater than or equal to one. Note in particular that $\mathcal{E}$ is not empty.

Let $i = 1$ or 2. Let $(S_i; \alpha^{(i)} := \bigcup_{j=1}^{g} \alpha_j^{(i)}, \beta^{(i)} := \bigcup_{j=1}^{g} \beta_j^{(i)}; D_i)$ be a punctured Heegaard diagram encoded by the E-datum $\mathcal{E}$. Since the P_1 and P_2 have the same neighborhood of their branched loci $S(P_1)$ and $S(P_1)$, and the branched loci of the flow-spines are subsets of the union $\alpha^{(i)} \cup \beta^{(i)}$ of slopes by the construction of flow-spines from punctured Heegaard diagrams as explained in Section 2.2, we can say that the slopes $\alpha_{j_1}^{(1)}$ and $\alpha_{j_2}^{(2)}$ (resp. $\beta_{j_1}^{(1)}$ and $\beta_{j_2}^{(2)}$) *share* the corresponding subsets, which are immersed subcurve, of the branched loci. Then we may assume that $\alpha_j^{(1)}$ and $\alpha_j^{(2)}$ (resp. $\beta_j^{(1)}$ and $\beta_j^{(2)}$) do so, up to reordering.

The Whitehead graph $W_\alpha^{(i)}$ (with the obvious embedding into S^2) is defined as follows. Let $Q_\alpha^{(i)}$ be the closure of $S_i \setminus \alpha^{(i)}$ in the path metric. Thus $Q_\alpha^{(i)}$ is the g-th punctured 2-sphere. Every arc of $\beta^{(i)} \cap Q_\alpha^{(i)}$ belongs to one of finitely many homotopy classes of properly embedded arcs in $Q_\alpha^{(i)}$. The vertices of $W_\alpha^{(i)}$ are the boundary components of $Q_\alpha^{(i)}$. For every homotopy class of arcs $\beta^{(i)} \cap Q_\alpha^{(i)}$ we have an edge with the obvious endpoints. Each slope $\alpha_j^{(i)}$ gives rise to two boundary components $\alpha_{j+}^{(i)}$ and $\alpha_{j-}^{(i)}$ of $Q_\alpha^{(i)}$ (the sign can be defined for all slopes in a consistent manner since the surface S_i and the slopes are oriented), then we denote the vertices of $W_\alpha^{(i)}$ corresponding to them by $u_j^{(i)}$ and $v_j^{(i)}$, respectively.

Note that there is a region in S^2 (where the graph $W_\alpha^{(i)}$ is embedded) corresponding to the joining disk $D^{(i)}$, which is bounded by g vertices $u_1^{(i)}, \dots, u_g^{(i)}$ (say) and g edges $e_1^{(i)}, \dots, e_g^{(i)}$.

Note that $W_\alpha^{(1)}$ and $W_\alpha^{(2)}$ has the same number of components, since the two punctured Heegaard diagrams has the same E-datum. In fact, the two vertices u and v of $W_\alpha^{(i)}$ are connected by an edge if and only if

(1) $u = u_j^{(i)}$ and $v = u_{j+1}^{(i)}$ or $v = u_j^{(i)}$ and $u = u_{j+1}^{(i)}$ (subscripts mod g); or
(2) $\alpha_{j_1+}^{(i)}$ (or $\alpha_{j_1-}^{(i)}$) and $\alpha_{j_2+}^{(i)}$ (or $\alpha_{j_2-}^{(i)}$) are connected by a subarc of $\beta \cap Q_\alpha^{(i)}$ which is a subset of the branched locus of P_i.

Let $W_\alpha^{(i)} = U_0^{(i)} \cup U_1^{(i)} \cup \cdots \cup U_N^{(i)}$ be the decomposition into connected components. We may assume without loss of generality that $U_0^{(i)}$ contains the vertices $u_1^{(i)}, \dots, u_g^{(i)}$. Take a vertex $v_{i_k}^{(i)}$ of $U_k^{(i)}$ for each $1 \leqslant k \leqslant N$.

The proof is an induction on the block number g. The case where $g \leqslant 1$ follows from [7]. Suppose $g \geqslant 2$.

Let $F^{(i)}$ be the union of all non-disk components of $\overline{S_i \setminus N(\alpha^{(i)} \cap \beta^{(i)})}$. Since there are exactly N simple closed curves on $Q_\alpha^{(i)}$ which separate the components of $W_\alpha^{(i)}$ from each other, we have $n(F^{(1)}) = n(F^{(2)}) = N$. If $N = 0$, then $W_\alpha^{(i)}$ is connected and thus every component of the closure of $S_i \setminus (\alpha^{(i)} \cap \beta^{(i)})$ in the path metric is a closed disk. This means that the flow spine P_i is standard by the construction. Therefore we obtain the desired result, that is, $M_1 \cong M_2$, due to [2] and [4]. Assume that $N > 0$. Let $C_1^{(i)}, \dots, C_N^{(i)}$ be a boundary separating system of $F^{(i)}$. Then by Lemma 4.2, M_i can be decomposed as $M_i' \# (\#_N (S^2 \times S^1))$ for some closed 3-manifold M_i' (possibly $M_i' \cong S^3$), and M_i' has a Heegaard diagram $(S_i'; \bigcup_{j \in I} \alpha_j, \bigcup_{j \in J} \beta_j)$, where S_i' is obtained from S_i by compressing along $C_1^{(i)}, \dots, C_N^{(i)}$ and $I = \{1, \dots, g\} \setminus \{i_1, \dots, i_N\}$ (resp. $J = \{1, \dots, g\} \setminus \{j_1, \dots, j_N\}$).

Now, we may produce a new joining disk D_i' for the Heegaard diagram $(S_i'; \bigcup_{j \in I} \alpha_j^{(i)}, \bigcup_{j \in J} \beta_j^{(i)})$ by modifying algorithmically the slopes α in such a way that the new punctured Heegaard diagrams $(S_1'; \bigcup_{j \in I} \alpha_j^{(1)}, \bigcup_{j \in J} \beta_j^{(1)}; D_1')$ and $(S_2'; \bigcup_{j \in I} \alpha_j^{(2)}, \bigcup_{j \in J} \beta_j^{(2)}; D_2')$ have the same E-data $\mathcal{E}'$. See Example 4 which shows the algorithm to produce a new joining disk. Since $bl(\mathcal{E}') \leqslant g(S_i') < g(S_i) = g$, recall the argument in Section 2.2, the proof is completed due to the inductive assumption.

Example. Consider the case where $g = 6$, $I = \{1, 2, 5\}$ and $J = \{1, 3, 4\}$ in the above proof.

Let $\lambda_1 := (\partial \overline{D_i} \cap \beta_1^{(i)}) \cup (\alpha_2^{(i)} \setminus \partial \overline{D_i}) \cup (\partial \overline{D_i} \cap (\beta_2^{(i)} \cup \alpha_3^{(i)} \cup \beta_3^{(i)} \cup \alpha_4^{(i)}))$, $\lambda_2 := \partial \overline{D_i} \cap (\beta_2^{(i)} \cup \alpha_3^{(i)})$ and $\lambda_5 := \partial \overline{D_i} \cap (\beta_5^{(i)} \cup \alpha_6^{(i)} \cup \beta_6^{(i)} \cup \alpha_1^{(i)})$. Shift $\lambda_1^{(i)}$ slightly to the outward of D_i and $\lambda_2^{(i)}$ and $\lambda_5^{(i)}$ slightly to the inward of D_i. Then by modifying $\alpha_1^{(i)}$, $\alpha_2^{(i)}$ and $\alpha_5^{(i)}$ along $\lambda_1^{(i)}$, $\lambda_2^{(i)}$ and $\lambda_5^{(i)}$, respectively, as shown in Fig. 9, we get the new joining disk D_i'. Note that the E-data corresponding the resulting punctured Heegaard diagrams for $i = 1, 2$ are obviously the same.

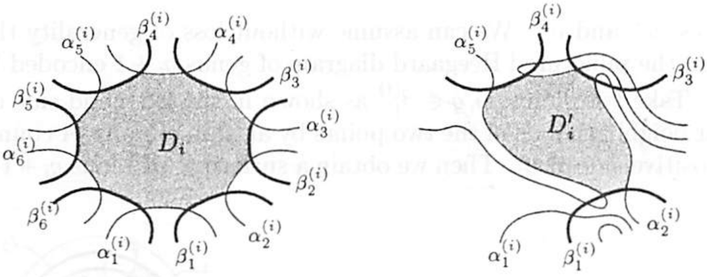


FIGURE 9. The deformation of $\beta_{i_0-1}^{(i)}$ and $\beta_{i_0}^{(i)}$

5. REDUCIBILITY AND CONNECTED SUMS

Using punctured Heegaard diagrams, we obtain an E-datum of the connected sum. We believe that these result is quite interesting for algorithmic and computer study of closed oriented 3-manifolds.

Proof of Theorem 1.3. For the proof of 1 and 2, we prove the former one, that is, the statement for $p_1^{-l}p_2^{-r}\mathcal{E}_1p_2^{+r}q_1^{+l}q_1^{-l}q_2^{-r}\mathcal{E}_2q_2^{+r}p_1^{+l}$.

1. Assume first that both $\mathcal{E}_1$ and $\mathcal{E}_2$ are negatively normalized. Let $(S_i, \alpha^{(i)} := \bigcup_j \alpha_j^{(i)}, \beta^{(i)} := \bigcup_j \beta_j^{(i)}; D_i)$ be a punctured Heegaard diagram of genus g_i corresponding to the E-datum $\mathcal{E}_i$ for $i = 1, 2$. Let S be the connected sum of S_1 and S_2 obtained by deleting the interior of closed disks B_i in $\mathrm{Int}D_i$ and attaching $S_1 \setminus B_1$ to $S_2 \setminus B_2$ by an arbitrary homeomorphism $h : \partial B_1 \to \partial B_2$. Then $(S; \alpha := (\bigcup_j \alpha_j^{(1)}) \cup (\bigcup_j \alpha_j^{(2)}), \beta := (\bigcup_j \beta_j^{(1)}) \cup (\bigcup_j \beta_j^{(2)}))$ is a Heegaard diagram of $M_1\#M_2$. Let λ_1, λ_2 be mutually disjoint simple arcs on S such that λ_1 connects $\beta_{g_1}^{(1)}$ to $\alpha_1^{(2)}$, λ_2 connects $\beta_{g_2}^{(2)}$ to $\alpha_1^{(1)}$ and $(\mathrm{Int}\lambda_i) \cap (\alpha \cup \beta) = \emptyset$ for $i = 1, 2$. Deform the slopes $\beta_{g_i}^{(i)}$ along λ_i by isotopy as in the proof of Theorem 1.1, see Fig. 10. Then the resulting Heegaard diagram has a joining disk D shown in Fig.

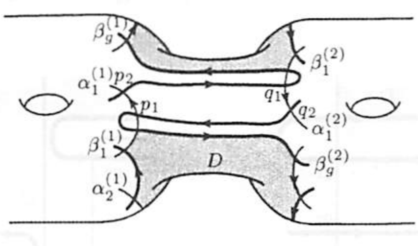


FIGURE 10. A joining disk of a Heegaard diagram of $M_1\#M_2$

10 and hence $(S; \alpha, \beta; D)$ is a punctured Heegaard diagram whose underlying 3-manifold is M. Take the points p_1, p_2, q_1, q_2 as shown in the figure. Then an E-datum encoding $(S; \alpha, \beta; D)$ is $p_1^{-l}p_2^{-r}\mathcal{E}_1p_2^{+r}q_1^{+l}q_1^{-l}q_2^{-r}\mathcal{E}_2q_2^{+r}p_1^{+l}$.

We now prove the general case, i.e. the case where $\mathcal{E}_1$ and $\mathcal{E}_2$ may not be negatively normalized. Let $i = 1$ or 2. Let $bl(\mathcal{E}_i) = g_i$ and $(S_i, \alpha^{(i)} := \bigcup_j \alpha_j^{(i)}, \beta^{(i)} := \bigcup_j \beta_j^{(i)}; D_i)$ be a punctured Heegaard diagram of genus g_i encoded by $\mathcal{E}_i$. Let $\mathcal{E}_i$ start and end with positive vertices, u^+ and v^+. We can assume without loss of generality that $u, v \in \beta_1^{(i)}$. Then we obtain the punctured Heegaard diagram of genus $g_i + 1$ encoded by $\mathcal{E}_i$ as in the following way. Take two points $p, q \in \beta_1^{(i)}$ as shown in the left-hand side of Fig. 11 and replace regular neighborhoods of the two points by an thin annulus A connecting p and q through the positive side of S_i. Then we obtain a surface S'_i of genus $g_i + 1$. Let γ denote

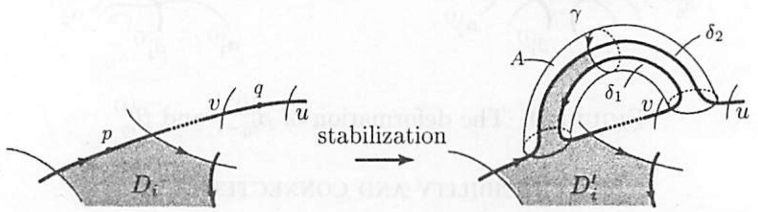


FIGURE 11

the core slope of A and δ_1, δ_2 denote the two slopes shown in the right-hand side of Fig. 11. Then $(S'_i; \alpha'^{(i)} := \bigcup_j^{g_1+1} {\alpha'_j}^{(i)}, \beta'^{(i)} := \bigcup_j^{g_1+1} {\beta'_j}^{(i)}; D'_i)$ is a punctured Heegaard diagram of genus $g_i + 1$, where ${\alpha'_1}^{(i)} := \gamma$, ${\alpha'_j}^{(i)} := {\alpha_{j-1}}^{(i)}$ (for $2 \leqslant j \leqslant g_i + 1$), ${\beta'_1}^{(i)} := \delta_1$, ${\beta'_j}^{(i)} := {\beta_{j-1}}^{(i)}$ (for $2 \leqslant j \leqslant g_i$), ${\beta'_{g_1+1}}^{(i)} := \delta_2$ and D'_i is naturally found as shown in the figure. It is clear that the E-datum of the resulting punctured Heegaard diagram (read by starting at ${\beta'_{g_1+1}}^{(i)} \cap {\alpha'_1}^{(i)}$) is $\mathcal{E}_i$. (We call this operation a *stabilization* of a punctured Heegaard diagram and explained it in [16].) In the case where $\mathcal{E}_i$ starts and ends with negative vertices, we can apply almost the same argument and obtain a new punctured Heegaard diagram $(S'_i; \alpha'^{(i)}, \beta'^{(i)}; D'_i)$ of genus $g_i + 1$ such that the E-datum read by starting at the unique point ${\beta'_{g_1+1}}^{(i)} \cap {\alpha'_1}^{(i)}$ is $\mathcal{E}_i$.

Then we can apply the same argument as in the negatively normalized case to the punctured Heegaard diagrams of one more higher genus constructed as above, if necessary, we get the assertion.

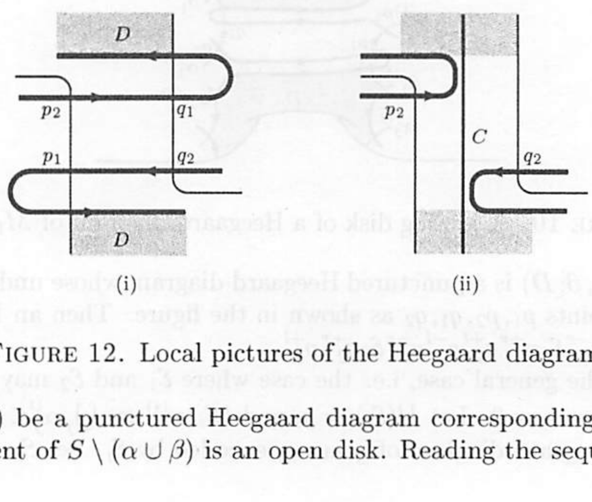


FIGURE 12. Local pictures of the Heegaard diagrams

2. Let $(S; \alpha, \beta; D)$ be a punctured Heegaard diagram corresponding to $\mathcal{E}$. By Lemma 4.2, each component of $S \setminus (\alpha \cup \beta)$ is an open disk. Reading the sequence $\mathcal{E}$, one easily

sees that a part of $(S;\alpha,\beta;D)$ can be depicted as in Fig. 12 (i) (in the figure, the system α is drawn by thin lines and β is drawn by bold lines). In this case, we can locally isotope β so that there exists an essential simple closed curve C disjoint from $\alpha\cap\beta$ as shown in Fig. 12 (ii). If C is non-separating in S, then the manifold M admits an embedded non-separating 2-sphere, and it contradicts the assumption. Therefore C is separating in S and the compression of S along C produces two Heegaard diagrams of closed orientable 3-manifolds. The two diagrams have the natural joining disks and the E-data encoding the diagrams are $\mathcal{E}_1$ and $\mathcal{E}_2$. (Recall that the above operation is local.)

Now, the assertion is an immediate consequence of Theorems 1.2 and 1.3.1.

Remark that if we consider the above theorem in the category of *standard* flow-spines, we do not need the condition that M does not contain an embedded non-separating 2-sphere. Fig. 13 shows the local replacement of closed normal o-graphs corresponding to Theorem 1.3.

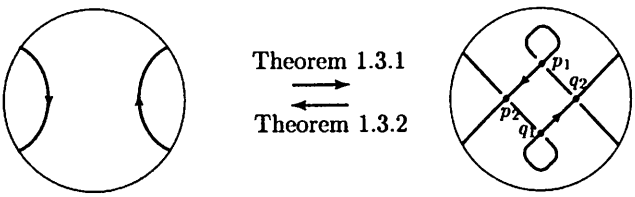


FIGURE 13. Connected sum and its inverse of closed normal o-graphs

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