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Heegaard-type presentations of branched spines and Reidemeister-Turaev torsion

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Abstract

Reidemeister-Turaev torsion is an invariant of a 3-manifold M equipped with a Spin^c-structure, one representation of which is a homology class of a non-singular vector field on M. In this paper, we introduce a way to represent a branched spine, which can be regarded as a combinatorial presentation of a Spin^c-structure on a 3-manifold, as a Heegaard diagrams with a *joining system*, and explain an accessible way to compute the invariant using this presentation.

Introduction

The theory of *branched spines* of 3-manifolds was introduced by Benedetti and Petronio in [2] as a combination of the two classical concepts of *simple spine* and *branched surface* for combinatorial encodings of 3-manifolds with non-singular vector field. Notice that a special class of them, which is called a *flow-spine*, was, in fact, formerly introduced by Ishii in [7] and it plays an important role in the study of closed 3-manifolds with non-singular vector field, see also [2, Chapter 5-6].

A branched spine P of a 3-manifold M is represented as a faceted surface together with an orientation-reversing face-pairing ϕ on it by cutting M along P and watching its section, see [2, 6] for details. In this paper, we introduce a new way to represent branched spines, *punctured Heegaard diagram*, which is defined in a similar mannar as a Heegaard diagram of a compact 3-manifold. In the case when P is a flow-spine of a closed 3-manifold M, the corresponding punctured Heegaard diagram is, indeed, a Heegaard diagram of M with an extra information, which we call a *joining disk.*

In [12], Turaev refined the Reidemeister torsion as an invariant of manifold equipped with Spin^c-structure, the invariant is called the *Reidemeister-Turaev torsion.* Turaev's reformulation of Spin^c-structure allows us to regard it as a *homology class* of non-singular vector fields, and Benedetti and Petronio [3] introduced a way to calculate the Reidemeister-Turaev torsion of the Spin^c-structure represented by a branched spine. In [9], we developed their way to compute Reidemeister-Turaev torsion in particular when P is a flow-spine using the Heegaard splitting which a flow-spine naturally induces. In the present paper, this idea is explained in terms of punctured Heegaard diagrams and it provides a quite easy way to calculate the invariant.

We introduce a calculus for punctured Heegaard diagrams corresponding to *regular moves* for flow-spines, see [8]. A new move, which punctured Heegaad diagrams naturally carry, is also discussed.

1 Reidemeister-Turaev torsion

Let F be a field and let E be a n-dimensional vector space over F. For two ordered bases $b = (b_1, \ldots, b_n)$ and $c = (c_1, \ldots, c_n)$ of F, we write $[b/c] = \det(a_{ij}) \in F^\times$, where $b_i = \sum_{j=1}^n a_{ij}c_j$. The bases b and c are said to be *equivalent* if $[\mathrm{b}/\mathrm{c}] = 1$.

Let $C = (0 \xrightarrow{\partial_m} C_m \xrightarrow{\partial_{m-1}} C_{m-1} \to \cdots \to C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\partial_{-1}} 0)$ be a finite dimensional chain complex over F. For each $0 \le i \le m$, set $B_i = \mathrm{Im}\ \partial_i$, $Z_i = \mathrm{Ker}\ \partial_{i-1}$ and $H_i = Z_i/B_i$. The chain complex is said to be *acyclic* if $H_i = 0$ for all i. Suppose that C is acyclic and C_i is endowed with a ***distinguished*** basis c_i for each i. Choose an ordered set of vectors b_i in C_i for each $i = 0, \ldots, m$ such that $\partial_{i-1}(b_i)$ forms a basis of B_{i-1}. By the above construction, $\partial_i(b_{i+1})$ and b_i are combined to be a new basis $\partial_i(b_{i+1})b_i$ of C_i. With this notation, the *torsion* of C is defined by

$$\tau(C) := \prod_{i=0}^{m} [\partial_i(b_{i+1})b_i/c_i]^{(-1)^{i+1}} \in F^\times.$$

Let M be a compact connected orientable smooth manifold of an arbitrary dimension. Let X be a CW-decomposition of M, $\hat{X} \to X$ be its maximal abelian cover and F be a field. We can equip $\hat{X}$ with the CW-structure naturally induced by that of X, and then we regard $C_*(\hat{X})$ as a left $\mathbb{Z}[\pi_1(X)]$-module via the monodromy.

Let $\{e_i^k\}$ be the set of all oriented k-cells in X. When we choose a family $\{\hat{e}_i^k\}$ of cells of $\hat{X}$ and orient and order these cells in arbitrary way, it becomes a free $\mathbb{Z}[H_1(X)]$-basis of $C_k(\hat{X})$. In this way, we can regard $C_*(\hat{X})$ as a chain complex with basis.

Let $\varphi : \mathbb{Z}[\pi_1(X)] \to F$ be a ring homomorphism. If the based chain complex $C_*^\varphi(X) = F \otimes_\varphi C_*(\hat{X})$ over F is acyclic, the *Reidemeister torsion* of X is

$$\tau^\varphi(M) := \tau(C_*^\varphi(X)) \in F^\times / \pm\varphi(H_1(M)).$$

Otherwise, set $\tau^\varphi(M) = 0 \in F$.

Let M be a closed smooth 3-manifold. Two non-singular vector fields $\mathcal{V}_1, \mathcal{V}_2$ on M are called *homologous* if there exists a closed 3-ball $B \subset M$ such that the restrictions of $\mathcal{V}_1, \mathcal{V}_2$ to $M \backslash \mathrm{Int}(B)$ are homotopic as non-singular vector fields. A Spin^c-*structure* is a homology class of non-singular vector fields. We denote by $\mathrm{Spin}^c(M)$ the set of Spin^c-structures on M. The action of $H_1(M)$ to $\mathrm{Spin}^c(M)$ is understood using the obstruction theory, see [12]. Turaev's idea is that each Spin^c-structure $[\mathcal{V}]$ determines the basis (as a set) of $C^\varphi(M)$. Then we can define the torsion $\tau^\varphi(M, [\mathcal{V}], \mathfrak{o}_M) \in F^\times$ of homologically oriented 3-manifold equipped with Spin^c-structure, which we call the *Reidemeister-Turaev torsion*, see [3].

2 Heegaard diagrams and punctured Heegaard diagrams

By a *Heegaard diagram* we means a triple $(S;\alpha,\beta)$ where S is a closed, connected, orientable surface and $\alpha = \bigcup_{i=1}^{m} \alpha_i$ and $\beta = \bigcup_{i=1}^{n} \beta_i$ are compact 1-manifolds on S.

A Heegaard diagram gives rise to a 3-manifold

$$M = M(S;\alpha,\beta) := S \times [-1,1] \bigcup_{\alpha\times\{-1\}} (\text{2-handles}) \bigcup_{\beta\times\{1\}} (\text{2-handles})$$

obtained by adding 2-handles $H_{\alpha_1},\dots,H_{\alpha_m}$ and $H_{\beta_1},\dots,H_{\beta_n}$ to $S\times[-1,1]$ along the curves $\alpha_1\times\{-1\},\dots,\alpha_m\times\{-1\}$ and $\beta_1\times\{-1\},\dots,\beta_m\times\{-1\}$, respectively. We denote $\overline{M}$ the manifold obtained by adding 3-handles along all resulting 2-sphere boundary components of M. The decomposition of M or $\overline{M}$ by $S\times\{0\}$ is the associated Heegaard splitting of M and the genus of S is called the *genus* of the splitting. We will denote the core of H_{α_i} (resp. H_{β_i}) by D_{α_i} (resp. D_{β_i}).

A Heegaard diagram is said to be *ordered* if the components of α and β are ordered, respectively. An *oriented* Heegaard diagram is an oriented triad $(S;\alpha,\beta)$, and determines an oriented splitting by the convention that the positive normal to S in M points toward the β-side of the splitting. An oriented Heegaard diagram $(S;\alpha,\beta)$ with a fixed point $p_i\in\beta_i\backslash\alpha$ for each β_i is said to be *based*. We often denote an ordered, based Heegaard diagram by $(S;\{\alpha_i\}_{i=1}^m,\{\beta_j\}_{j=1}^n;\{p_k\}_{k=1}^n)$. A system of pairwise disjoint oriented simple closed curves $\gamma=\bigcup_{i=1}^n\gamma_i$ on S is called a *dual system* of β if it satisfies that γ_i intersects β_i transversely once at the point p_i in the positive direction shown in Figure 1 and $\gamma_i\cap\beta_j=\emptyset$ when $i\neq j$.

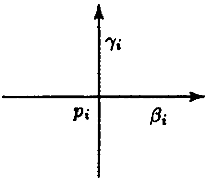

Figure 1: The positive intersection with a dual loop

Given an oriented Heegaard diagram $(S;\alpha,\beta)$. Let $J=\bigcup_{i=1}^{l}J_i$ be a disjoint union of components of $S\backslash(\alpha\cup\beta)$. Then it is said to be *joining* if $\partial\overline{J}$ is a disjoint union of simple loops and $\partial J\cup\beta_j$ is an arc for $j=1,\dots,g$ on which point J lies on the right-side. In the case when $l=1$ and J_1 is a disk, we call J_1 a *joining disk.* Figure 11 illustrates examples of joining disks.

REMARK 2.1. If we find a disjoint union $J=\bigcup_{i=1}^{l}J_i$ of components of $S\backslash(\alpha\cup\beta)$ such that ∂J is a disjoint union of simple loops, $\partial J\cup\beta_j$ is an arc for $j=1,\dots,g$ for an *non-oriented* Heegaard diagram $(S;\alpha,\beta)$, we can orient the diagram to make J joining. we call J a joining union for the non-oriented Heegaard diagram.

We call a(n) (oriented) Heegaard diagram $(S;\alpha,\beta)$ with joining union J a *punctured Heegaard diagram* and denote it by $(S;\alpha,\beta;J)$.

A punctured Heegaard diagram gives rise to a 3-manifold $M = M(S;\alpha,\beta;J)$ obtained by adding 2-handles $H_{\alpha_1},\dots,H_{\alpha_m}$ and $H_{\beta_1},\dots,H_{\beta_n}$ to $(S\backslash J)\times[-1,1]$ along the curves $\alpha_1\times\{-1\},\dots,\alpha_m\times\{-1\}$ and $\beta_1\times\{-1\},\dots,\beta_m\times\{-1\}$, respectively. We denote $\overline{M}$ the manifold obtained by adding 3-handles along all resulting 2-sphere boundary components of M.

LEMMA 2.2. *Let D be a joining disk of a Heegaard diagram $(S;\alpha,\beta)$ and α and β respectively bound complete systems of meridian disks of the closures of the corresponding components of $\overline{M(S;\alpha,\beta)\backslash S}$. Then $\overline{M(S;\alpha,\beta)} = \overline{M(S;\alpha,\beta;D)}$.*

PROOF. It is clear from the fact that taking a connected sum with the 2-sphere does not change the homeomorphism type of a surface. □

3 A combinatorial computation of Reidemeister-Turaev torsion

Given a ordered, based Heegaard diagram $(S;\{\alpha_i\}_{i=1}^g,\{\beta_j\}_{j=1}^g;\{p_k\}_{k=1}^g)$ of genus g of a closed 3-manifold M. Then D_{α_i} determines an element $x_i\in\pi_1(M,*)$ and β_j determines $r_j = r_j(x_1,\dots,x_g)\in\pi_1(M,*)$ starting at the point p_j and following the oriented loop β_j, for each $i,j=1,\dots,g$. Moreover, if we choose a dual system $\{\gamma_1,\dots,\gamma_g\}$ of $\{\beta_1,\dots,\beta_g\}$, γ_i determines $y_j\in\pi_1(M,*)$ in the same manner. Let $p:\mathbb{Z}[\pi_1(M)]\to\mathbb{Z}[H_1(M)]$ be the canonical projection and denote $[z]=p(z)$ for $z\in\pi_1(M)$.

PROPOSITION 3.1. *Let $\varphi:\mathbb{Z}[H_1]\to F$: be a ring homomorphism and*

$$0\to C_3^{\varphi}(M)\xrightarrow{\partial_2}C_2^{\varphi}(M)\xrightarrow{\partial_1}C_1^{\varphi}(M)\xrightarrow{\partial_0}C_0^{\varphi}(M)\to 0.$$

be an acyclic based finite chain comples over a field F. Then, for some base of C_*^{φ}, ∂_2^{φ}, ∂_1^{φ} and ∂_0^{φ} respectively have the following matrix presentations:

$$\begin{pmatrix}\varphi([y_1])-1\\ \vdots\\ \varphi([y_g])-1\end{pmatrix},\quad \left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1\geq i,j\geq g}\quad \textit{and}\quad \begin{pmatrix}\varphi([x_1])-1 & \cdots & \varphi([x_g])-1\end{pmatrix}.$$

COROLLARY 3.2. *Let the twisted chain complex $C_*^{\varphi}(M)$ be acyclic. Then there exist two integers $1\leq^{\exists} k,l\leq n$ such that*

$$\tau^{\varphi}(M,[\mathcal{V}]) = \frac{\det B_{k,l}}{(\varphi([x_k])-1)(\varphi([y_l])-1)}\in F^{\times},$$

where $B_{k,l}$ is the matrix obtained by removing k-th row and l-th column from the matrix $\left(\varphi\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{1\leq i,j\leq n}$. We call this value the Reidemeister-Turaev torsin of the ordered, marked Heegaard diagram $(S;\{\alpha_i\}_{i=1}^g,\{\beta_j\}_{j=1}^g;\{p_k\}_{k=1}^g)$.

4 Punctured Heegaard diagrams and branched standard spines

Let M be a compact orientable 3-manifold. $P \subset M$ is called a *branched surface* if P is locally modeled one of the following 3 models:

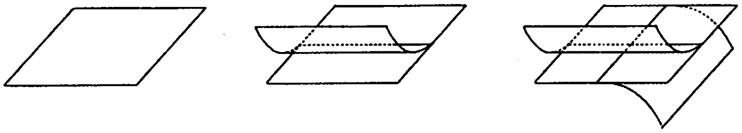

Figure 2: Local pictures of a branched surface

In this paper, all branched surfaces are assumed to be *transversely oriented*, this is the natural generalization of the 2-sidedness of embedded surfaces. Denote L a branched locus of P. We call a component of $P\backslash L$ a *face* of P. Let S be a face of P. If all branch directions along $\partial\overline{S}$ point out from S, $P\backslash S$ is still a branched surface.

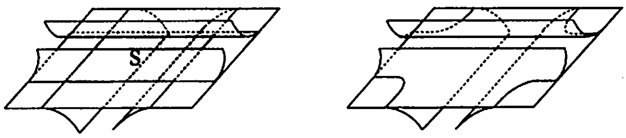

Figure 3: Removable face

One can regard $N(P)$ as a interval bundle over P. The boundary $\partial N(P)$ decomposes into two parts: the endpoints of the fibers, $\partial_h N(P)$, and the rest, $\partial_v N(P)$, see [5, 10] for more details about branched surfaces.

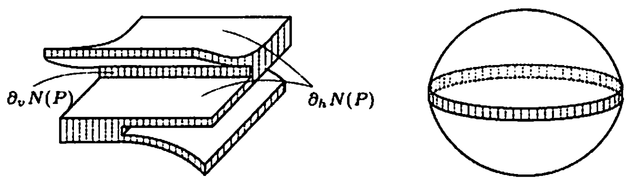

Figure 4: Regular neighborhood of a branched surface

P is called a *branched spine* if M collapses onto P in the case when $\partial M \neq \emptyset$. For a closed M, P is called *branched spine* if $M\backslash P$ is homeomorphic to the open 3-ball. A branched spine P is naturally stratified as $V(P) \subset S(P) \subset P$, where $V(P)$ is a set of *vertices* and $S(P)$ is the singular set. A branched spine P is called a *standard* if this stratification induces a CW decomposition of P (that is, there is no hoop, etc.), see [2] for a precise definition. A branched spine P is called a *flow-spine* if $\partial_v N(P)$ is an annulus. For a flow-spine of a closed 3-manifold, one can extend the I-fiber of the regular neighborhood of the spine to the ambient 3-manifold. In this way, a(n) oriented flow-spine P, determines a vector flow $\mathcal{V}(P)$ up to homotopy of non-singular vector flows $\mathcal{V}$ on the manifold.

THEOREM 4.1 (ISHII [7]). *Every homotopy class of non-singular vector flows is carried by a flow-spine.*

For each oriented Heegaard diagram $(S;\alpha,\beta)$ of the manifold M, we can associate an oriented branch structure with $B := S \cup (\bigcup D_{\alpha_i}) \cup (\bigcup D_{\beta_j})$ so that at each point on $\alpha \cup \beta$, the branch direction points leftword along the loop $\alpha \cup \beta$, see Figure 5.

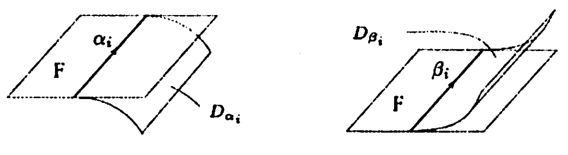

Figure 5: Branch direction around α_i and β_j

Let J be a joining union of $(S;\alpha,\beta)$. Then J is a removable faces of a branched surface B, and hence, $P = P(S;\alpha,\beta;J) := B\backslash J$ is a branched surface. The following lemma is clear from the above construction.

LEMMA 4.2. *P is a branched spine of $M(S;\alpha,\beta;J)$.*

THEOREM 4.3. *Every branched standard spine is obtained from a punctured Heegaard diagram in the above way. moreover, every flow-spine is obtained from a Heegaard diagram with joining disk.*

PROOF. Let P be a standard spine of a compact, oriented 3-manifold with boundary M. We may identify a collar neighborhood of ∂M with $\partial M \times [-1,0]$ ($\partial M \times \{0\} = \partial M$). Then the collapse induces an essentially unique retraction $\pi : M\backslash(\partial M \times (-1,0]) \to P$ such that $M\backslash(\partial M \times (-1,0])$ is the mapping cylinder of $\pi|_{\partial M \times \{-1\}}$. Then on the (possibly disconnected) surface ∂M, the 3-regular graph $\pi^{-1}(S(P))$, together with the disjoint union $e = e_1 \cup \cdots \cup e_r \subset \pi^{-1}(S(P))$ of circles corresponding to the annuli $\partial_v N(P)$, encodes the branched standard spine P, see [2, Page 28] for more detail. Note that we can identify M with $(\partial M \times [-1,0]) / \sim$ and P with $(\partial M \times \{-1\}) / \sim$, where $(\pi(x),-1) \sim (\pi(y),-1)$.

The circles e separates ∂M into two classes, the *black* part B and the *white* one W. Draw a loop C_i in the neighborhood of e_i such that $C_i \cap e_i \neq \emptyset$, $C_i \cap \pi^{-1}(S(P)) \subset e_i$, C_i intersects e_i transversely and $C_i \approx e_i$ in the neighborhood of e_i. Let S_B (resp. S_W) be the closure of the union of components of $\partial M \backslash \bigcup_i C_i$ corresponding to the black area B (resp. the white area W). Then $\overline{S_B\backslash B}$ is a disjoint union of disks $D_1^+,\ldots,D_n^+$ identified by π with disks $D_1^-,\ldots,D_n^- \subset B$, respectively. Set $S = \partial(S_B \times [-1,0])/\sim$, $\alpha_i = \overline{(\partial D_i^+/\sim)} \subset S$ and $\alpha = \bigcup_i \alpha_i$. $S = \overline{\partial(S_B \times [-1,0])\backslash(S_B \times \{-1\})}$. Now, the remaining part $\overline{P\backslash\pi(S_B)} = \pi(\overline{S_B\backslash B})$ is a set of disks $D_1',\ldots,D_n'$. Set $\beta_j = \partial D_j'$ and $\beta = \bigcup_j \beta_j$.

Now we claim that $(S;\alpha,\beta;J)$ is a punctured Heegaard diagram presenting $P \subset M$. The fact that J is a joining system is clear from the above construction. In fact, each component of $\overline{C}_i \cup S_B$ is a subarc of some slop α_j and each of $\overline{C}_i \cup S_W$ is a subarc of some β_k. $\overline{(S \cup (\bigcup_i D_i) \cup (\bigcup_j D_j))\backslash J} = \overline{S\backslash J \cup (\bigcup_i D_i) \cup (\bigcup_j D_j)} \approx \pi(\overline{B}) = P$. $M(S;\alpha,\beta;J) = M$. □

COROLLARY 4.4. *For a closed 3-manifold, every homotopy class of non-singular vector flows is carried by a Heegaard diagram with joining disk. In particular, every* Spin^c*-structure is carried by a Heegaard diagram with joining disk.*

PROOF. This immediately follows from Theorem 4.3 and Theorem 4.1. □

We denote $\mathcal{V}(S;\alpha,\beta;D) := \mathcal{V}(P(S;\alpha,\beta;D))$.

EXAMPLE 4.5. Figure 6 (i) illustrates the unique black area (, which is homeomorphic to the disk) for a flow-spine of the 3-manifold M obtained by the 0-surgery along the figure-eight knot in S^3. The bold line in Figure 6 (ii) shows the curve C_1 on ∂B^3, and (iii) is the resultant Heegaard diagram. Actually, we can transform the left diagram (ii) (except the pale edges) into (iii) by isotopy.

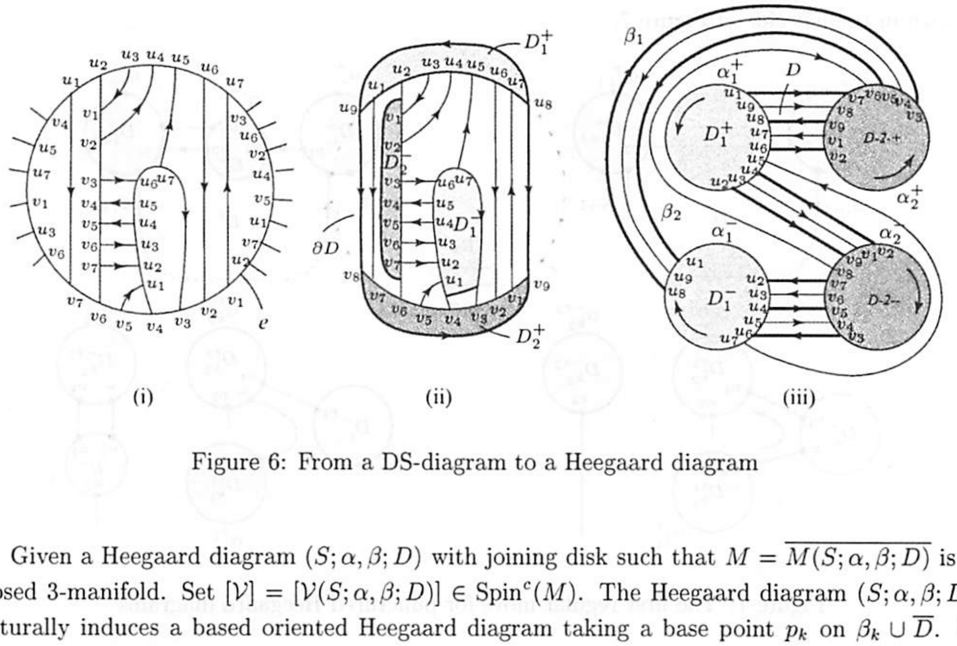

Figure 6: From a DS-diagram to a Heegaard diagram

Given a Heegaard diagram $(S;\alpha,\beta;D)$ with joining disk such that $M = \overline{M(S;\alpha,\beta;D)}$ is a closed 3-manifold. Set $[\mathcal{V}] = [\mathcal{V}(S;\alpha,\beta;D)] \in \mathrm{Spin}^c(M)$. The Heegaard diagram $(S;\alpha,\beta;D)$ naturally induces a based oriented Heegaard diagram taking a base point p_k on $\beta_k \cup \overline{D}$. It follows that the diagram induces the elements x_i's, r_i's and y_i's in $\pi_1(M)$ as in Section 2.

THEOREM 4.6. *Under the same assumption for φ, F, $C_*^\varphi(M)$ as in Proposition 3.1, the matrix presentations of the boundary operators ∂_2^φ, ∂_1^φ and ∂_0^φ with respect to a bases defined by the* Spin^c*-structure $[\mathcal{V}]$ are*

$$\begin{pmatrix} \varphi([y_1]) - 1 \\ \vdots \\ \varphi([y_g]) - 1 \end{pmatrix}, \quad \left(\varphi\left(\left[\frac{\partial r_j}{\partial x_i}\right]\right)\right)_{1\geq i,j\geq g} \quad \textit{and} \quad \begin{pmatrix} \varphi([x_1]) - 1 & \cdots & \varphi([x_g]) - 1 \end{pmatrix}.$$

PROOF. □

5 Calculus for oriented Heegaard splittings with joining removable disk

In this section, all punctured Heegaard diagram we consider are those with joining disk and corresponding to a closed 3-manifold. Recall that in this case, it is a Heegaard diagram of the closed 3-manifold after capping a disk to the punctured hole. We briefly introduce the calculus for oriented Heegaard diagrams with joining disk corresponding to the regular moves for flow-spines [8]. We also introduce a new move, which we call a *marking move*, which punctured Heegaad diagrams naturally carry.

5.1 first and second regular moves

Given a punctured Heegaard diagram $(S;\alpha,\beta;D)$ of genus g. Suppose that on the oriented circle α_i there exist successive vertices v_1 and v in this order such that $v_1 \in \overline{D}$ and $v \notin \overline{D}$ as drawn in the left side of Figure 7.

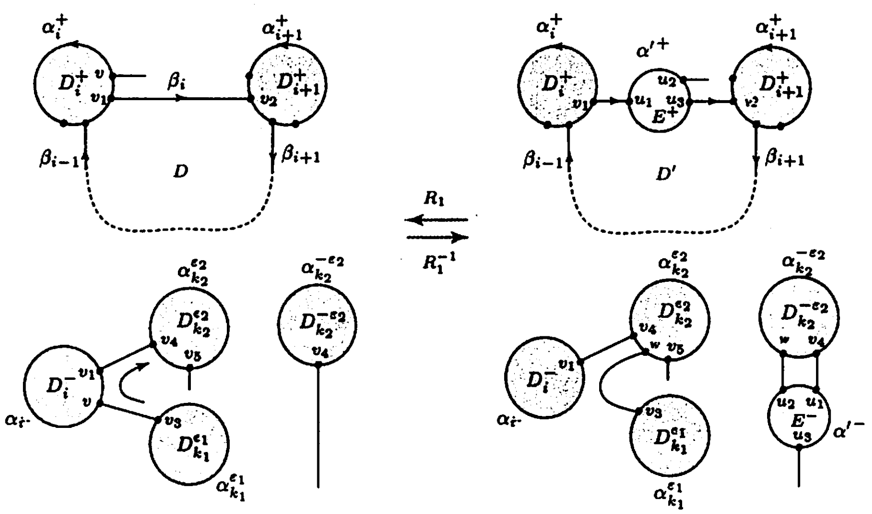

Figure 7: The first regular move for punctured Heegaard diagrams

Consider to change locally the diagram as shown in Figure 7. It is easy to check following the construction in Section 4 that this operation corresponds to the first regular move for flow spines and the region D' in the resulting diagram is a joining disk. We call the inverse of this operation a *first regular move.*

The operation given by the parallel argument by replacing the term α by β and inversing all edge directions are also called a *first regular move.*

Given a punctured Heegaard diagram $(S;\alpha,\beta;D)$. Let $(S;\alpha,\beta;D)$ have two edges $e_1 \subset \beta_i$ and e_2 which belong to the boundary of a region R and whose directions are coherent to the anti-clockwise orientation of ∂R, see the left side of Figure 8.

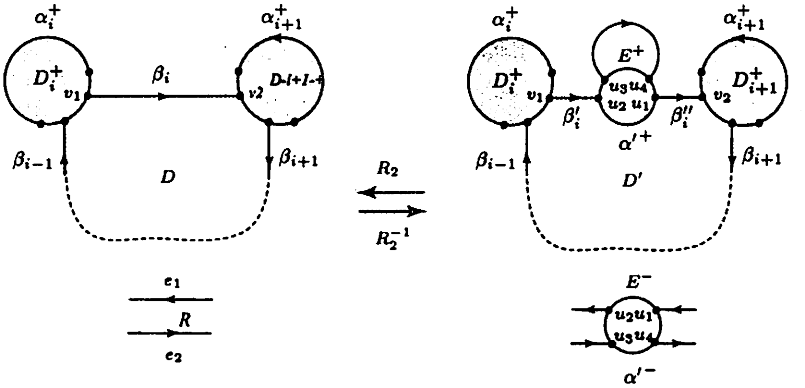

Figure 8: The second regular move for punctured Heegaard diagrams

Consider to change locally the diagram as shown in Figure 7. It is easy to check that the region D' in the figure becomes a joining disk. We call the inverse of this operation a *second regular move.*

The operation given by the parallel argument by replacing the term α by β and inversing all edge directions are also called a *second regular move.*

COROLLARY 5.1. *Let M be a closed 3-manifold equipped with a vector flow $\mathcal{V}$. Let $(S;\alpha,\beta;D)$ and $(S';\alpha',\beta';D')$ be two punctured Heegaard diagrams carrying $\mathcal{V}$. Then $(S';\alpha',\beta';D')$ is obtained from $(S;\alpha,\beta;D)$ by successive applications of the first and the second regular moves.*

PROOF. Since the above operations correspond to the first and second regular moves for flow-spines, this theorem directly follows from [8, Theorem 2.3]. ☐

5.2 marking move

Given a (non-oriented) Heegaard diagram $(S;\alpha,\beta)$ of genus g. Let D and D' be its joining disks. The disk D (resp. D') induces orientations o_{α_i} and o_{β_j} (resp. o'_{α_i} and o'_{β_j}) of the slopes α_i and β_j $(1 \leq i,j \leq g)$ so that D (resp. D') is removable, recall Remark 2.1. Let λ_{α_i} (resp. λ_{β_i}) denote an oriented loop which intersects $P = S;\alpha,\beta;D)$ transversely at a single point in D_{α_i})(resp. D_{β_i}) and $\lambda_{DD'}$ be an oriented loop which intersects P transversely at two points $p \in D$ and $p' \in D'$. The orientation of these loops are shown in Figure 9 and 10. Let $\{j_1,\ldots,j_m\} = \{n \mid o_{\alpha_i} \neq o'_{\alpha_i}\}$ and $\{k_1,\ldots,k_n\} = \{n \mid o_{\beta_i} \neq o'_{\beta_i}\}$.

DEFINITION 5.2. A *marking move* is the move from $(S;\alpha,\beta;D)$ to $(S;\alpha,\beta;D')$ where D and D' is joining disks of the (non-oriented) Heegaard diagram $(S;\alpha,\beta)$.

THEOREM 5.3. *The vector flow* $\mathcal{V}' = \mathcal{V}(S;\alpha,\beta;D')$ *is obtained from* $\mathcal{V} = \mathcal{V}(S;\alpha,\beta;D)$ *by a sequence of operations* $S'_{\lambda_{\alpha_{j_1}}}, \dots, S'_{\lambda_{\alpha_{j_m}}}, S_{\lambda_{\beta_{k_1}}}, \dots, S_{\lambda_{\beta_{k_n}}}, S_{\lambda_{DD'}}$, *where* $S'_{\lambda_{DD'}}$ *denotes the Reeb surgery along an oriented loop* $\lambda_{DD'}$ *and* $S_{\lambda_{DD'}}$ *denotes the Reeb surgery along* $\lambda_{DD'}$ *between two direction reversing operations of vector flows.*

PROOF. Recall that P is a flow-spine, to which the vector flow $\mathcal{V}(S;\alpha,\beta,D)$ is transversal. Though we can add D (resp. D') to P (resp. $P' = P(S;\alpha,\beta;D')$) so that the resulting is still a branched surface, (recall Figure 3), the vector flow $\mathcal{V}$ (resp. $\mathcal{V}'$) no longer remains transversal to it.

We can assume by modifying the flow homotopically that the set of points on D to which $\mathcal{V}$ is tangent is a circle C, see the left side of Figure 9.

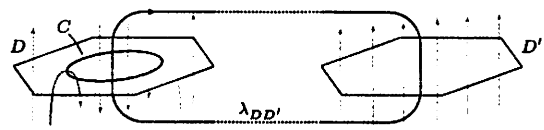

Figure 9: The loop $\lambda_{DD'}$

Let the branch directions of P and P' around the meridian disk D_{α_i} differ. Figure 10 (i) illustrates the branch direction and the vector flow $\mathcal{V}$ around D_{α_i}. The vector flow $\mathcal{V}$ has a circle tangency C_{α_i} on D_{α_i} to the branched surface P' as drawn in Figure 10 (ii).

Consider the vector flow $\mathcal{V}_1'$ obtained by applying the Reeb surgery along the orinented loop λ_{α_i} to the vector flow $-\mathcal{V}(S;\alpha,\beta,D)$, see Figure 10 (iii). Then $\mathcal{V}_1 = -\mathcal{V}_1'$ produces one more circle tangency C'_{α_i} on D_{α_i} to P' while the other parts remain unchanged. Now we can cancell the two circle tangencies C_{α_i} and C'_{α_i} by a homotopical deformation of the vector flow $\mathcal{V}_1$. Thus $\mathcal{V}$ and $\mathcal{V}_1$ intersect $P\backslash\mathrm{Int}(D_{\alpha_i})$ from the same side and D_{α_i} from different sides.

Let the branch directions of P and P' around the meridian disk D_{β_j} differ. Then the vector flow $\mathcal{V}$ intersects P' around the disk D'_{k_i} as in Figure 10 (iii). Thus we can apply the parallel argument as above without inversing the flow direction.

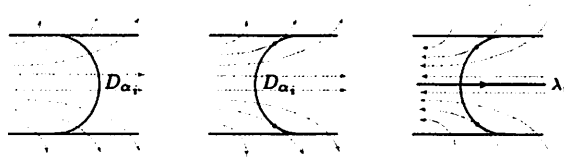

Figure 10: The flow around D_{α_i} and the oriented loop λ_{α_i}

After applying these operations to all D_{α_i}'s and D'_{β_i}'s around which the branch direction of P and P' differ, one obtains the flow $\mathcal{W}$ which is transversal to P' except on the circle C on D. Let $\mathcal{W}'$ be the vector flow obtained by applying the Reeb surgery to $\mathcal{W}$ along $\lambda_{DD'}$. Then $\mathcal{W}'$ and $\mathcal{V}'$ coincides in a neighborhood of the branched surface P', and they are homotopic on M since $M\backslash P'$ is an open 3-ball. □

Let ℓ_i (resp. ℓ'_i) $\in H_1(M)$ be represented by λ_i (resp. λ'_i) and $\ell_{DD'} \in H_1(M)$ be represented by $\lambda_{DD'}$.

COROLLARY 5.4. *The two* Spin^c*-structures* $[\mathcal{V}]$ *and* $[\mathcal{V}']$ *satisfies the following relation:*

$$[\mathcal{V}'] = [\ell_{DD'}] \cdot \prod_{i=1}^{m} [\ell_{\alpha_{j_i}}] \cdot c\left(\prod_{i=1}^{n} [\ell_{\beta_{k_i}}] \cdot c([\mathcal{V}]) \right),$$

where $c : \mathrm{Spin}^c(M) \to \mathrm{Spin}^c(M)$ *is a map defined by* $c([\mathcal{V}]) = [-\mathcal{V}]$.

PROOF. This follows from the above theorem and the definition of the action of $H_1(M)$ to Spin$^c(M)$. □

EXAMPLE 5.5. The Figure 11 illustrates a Heegaard diagram $(S; \alpha, \beta)$ of the manifold M in Example 4.5. It is easy to check that $[x_1] = [x_2]$ where $[x_1]$ and $[x_2]$ are the elements of $H_1(M)$ corresponding to α_1 and α_2, respectively, and thus $H_1(M) = \langle [x_1] \rangle \cong \mathbb{Z}$. Consider the two joining faces D, D' and D'' shown in the figure. Then the vector flow $\mathcal{V}(S; \alpha, \beta; D')$ is obtained from $\mathcal{V}(S; \alpha, \beta; D)$ by a sequence of three Reeb surgeries $R_{\lambda_{\beta_1}}$, $R_{\lambda_{\beta_2}}$ and $R_{\lambda_{DD'}}$. Moreover, we can check that $\ell'_1 = 0$, $\ell'_2 = [x_1]$ and $\ell_{DD'} = [x_1]$. Hence $[\mathcal{V}(S; \alpha, \beta; D')] = (\ell_{DD'} \ell_{\beta_1} \ell_{\beta_2}) \cdot [\mathcal{V}(S; \alpha, \beta; D)] = 2[x_1] \cdot [\mathcal{V}(S; \alpha, \beta; D)]$.

Similarly, the vector flow $\mathcal{V}(S; \alpha, \beta; D'')$ is obtained from $\mathcal{V}(S; \alpha, \beta; D)$ by a sequence of operations $R'_{\lambda_{\alpha_2}}$, $R_{\lambda_{\beta_1}}$, $R_{\lambda_{\beta_2}}$ and $R_{\lambda_{DD''}}$.

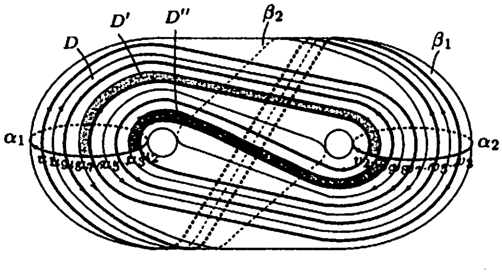

Figure 11: Joining disks D, D' and D'' on $(S; \alpha, \beta)$

QUESTION 5.6. For any two punctured Heegaard diagrams $(S; \alpha, \beta; D)$ and $(S; \alpha, \beta'; D')$, can one transfer one to the other by a successive application of first and second regular moves and marking moves?

6 Examples

6.1 Lens spaces

Let $L(p, q)$ be a lens space and $[\mathcal{V}_0]$ be the Spin^c-structure corresponding to a Seifert fibration of $L(p, q)$.

$\pi_1(L(p, q)) = \langle x \mid x^p \rangle$.

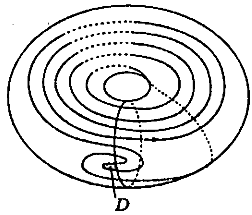

Figure 12: A punctured Heegaard diagram of the natural flow of $L(p,q)$

$\partial_2^\varphi = \left(\zeta^r - 1 \right), \partial_1^\varphi = 0, \partial_0^\varphi = \left(\zeta - 1 \right).$

Then we get

$$\tau^\varphi(L(p,q),[\mathcal{V}],\mathfrak{o}_{L(p,q)}) = \frac{1}{(\zeta-1)(\zeta^r-1)} \in \mathbb{C}.$$

In [11], Taniguchi, Tsuboi and Yamashita introduced an algorithm to obtain a DS-diagram with e-cycle starting from a Seifert data. Combining this result and our above construction, we have the following:

THEOREM 6.1. *There is an algorithm to obtain a punctured Heegaard diagram for a closed oriented Seifert fibered manifold M with a canonical vector flow starting from a given Seifert data.*

THEOREM 6.2. *We can compute the Reidemeister-Turaev torsion of a closed oriented Seifert fibered manifold $M(F;b,(p_1,q_1),\dots,(p_n,q_n))$ with a canonical vector flow in an algorithmic way.*

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