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A knot theory in the hyper-cubic graph Q_4

Kazuaki Kobayashi
Department of Mathematics, College of Arts and Sciences
Tokyo Woman's Christian University

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1 Introduction

Let G be a Hamilton graph (abbr. H-graph) and Δ be a Hamilton cycle (abbr. H-cycle) on G. If $(G, \Delta)(i = 1, 2)$ are such pair and if there is an automorphism $\varphi \in Aut(G)$ with $\varphi(\Delta_1) = \Delta_2$, we say that (G, Δ_2) belongs to the automorphism class $[G, \Delta_1]$ of (G, Δ_1). We denote the set of automorphism classes by $\{[G, \Delta]\}$. So $\{[G, \Delta]\} = \frac{\{(G,\Delta)\}}{Aut(G)}$. To investigate $\{[G, \Delta]\}$ we say a knot theory in the H-graph G. In this paper we will investigate a knot theory in the hyper cubic graph Q_4. In the sequel we denote $V(G), E(G)$ the sets of vertices, edges of G respectively. For other knot theory in other graphs, see ([K-2]).

2 Definition

Definition 1 *The hyper cubic graph Q_n is a graph defined as follows;*
(1) Q_n is the 1-skeleton of the n-cube $\underbrace{[0,1] \times \cdots \times [0,1]}_{n}$ *or*
(2) n times of the complete graph K_2 on 2 vertices, that is, inductively defines by

$$Q_1 = K_2$$
$$Q_n = Q_{n-1} \times K_2 \text{ (as a graph)}$$

Proposition 1 *The hyper cubic graph Q_n has the following properties;*
(1) Q_n is a n-regular, bipartite graph,

(2) $Q_n (n \geq 2)$ *is a Hamilton graph,*
(3) Q_n *does not contain the complete graph* K_3 *on 3 vertices and*
(4) The minimum sheet number of the slice presentation $ms(Q_n) = ms_\Delta(Q_n) = n-1$ *([K-H-T])*
(5) For $n \leq 3$ *it is a planar graph, and for* $n \geq 4$ *it is a non-planar graph.*
(6) The automorphism group of Q_n*,* $Aut(Q_n) = [S_2]^{S_n} \cong S_n \times (\mathbf{Z}_2)^n$ *([H]). So the order of the group is* $n! \times 2^n$*.*
(7) The genus $g(Q_n) = (n-4)2^{n-3} + 1$ *([B-H]).*
(8) The connectivity $\kappa(Q_n) = n$ *([K1]).*
(9) As it is a non planar graph, it is not adaptable by ([M-T]).
And Q_3 *is also non-adaptable proved by Yasuhara.*
(10) Q_4 *is a self-linked graph ([K-1]).*
(11) Q_5 *is a self-knotted graph proved by S.Suzuki.*

On more properties of Q_n and detail, see ([K-1])
To investigate $[Q_4, \Delta]$,we consider the complement $C(\Delta) := Q_4 - E(\Delta)$. Q_4 is a 4-regular and Δ is a cycle. So as $C(\Delta)$, there are following 7 possibilities;

(C_k is a cycle of length k and the cup is the disjoint union)

(1) C_{16} (2) $C_{12} \cup C_4$ (3) $C_{10} \cup C_6$ (4) $C_8 \cup C_8$
(5) $C_8 \cup C_4 \cup C_4$ (6) $C_6 \cup C_6 \cup C_4$ (7) $C_4 \cup C_4 \cup C_4 \cup C_4$

(Since Q_n is a bipartite graph, there are no cycles with odd length.)
To denote Q_4 diagramatically, we use a "square nested diagram (Fig.1)". And we say a square part (or square part edge), a floating edge as Figure 2.

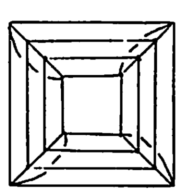


Fig. 1.

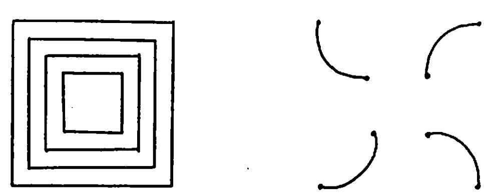


Fig. 2

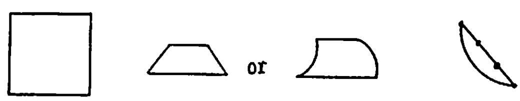
3 Propositions and Theorem

Proposition 2 *It does not happen that* $Q_4 - E(\Delta) = C_8 \cup C_4 \cup C_4$ *and* $\Delta \cap C_8$ *has no adjacent vertices on* Δ*.*

Proof. If $\Delta \cap C_8 = 8$ vertices do not adjacent on Δ, $\Delta \cap C_8$ are every other vertices on Δ. So they have same parity and C_8 is defined by joining the vertices with same parity. It is a contradiction. □

Proposition 3 *It does not happen the case* $\mathcal{C}(\Delta) = C_{12} \cup C_4$.

Proof. The patterns of 4-cycle contained in the square nested diagram of Q_4 are the following C_4^1, C_4^2, C_4^3 (Fig. 3). If $C(\Delta) = C_{12} \cup C_4$, C_{12} and C_4 are disjoint each other. So we delete C_4 and the edges incident to C_4 from Q_4.



C_4^1 C_4^2 C_4^3

Fig. 3

(1)The case $C_4 = C_4^1$. The patterns of 12-cycles contained in $Q_4 - C_4^1$ (Fig.4) are $C_{12}^{1,1}$, $C_{12}^{1,2}$, $C_{12}^{1,2}$ (Fig.5) (up to rotation and reversing of Q_4)

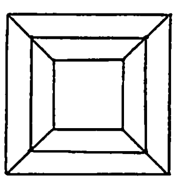
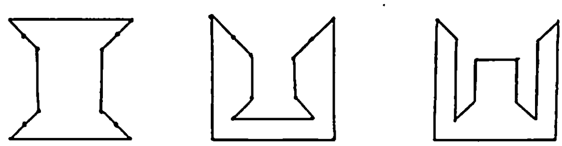


Fig. 4.



$C_{12}^{1,1}$ $C_{12}^{1,2}$ $C_{12}^{1,3}$

Fig. 5

(2)The case $C_4 = C_4^2$. The patterns of 12-cycles contained in $Q_4 - C_4^2$ (Fig.6) are $C_{12}^{2,1}$, $C_{12}^{2,2}$, $C_{12}^{2,3}$ (Fig.7) (up to rotation and reversing of Q_4)

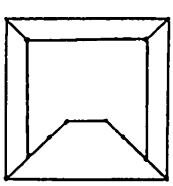
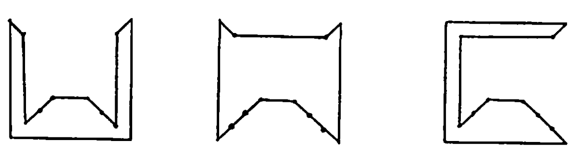


Fig. 6



$C_{12}^{2,1}$ $C_{12}^{2,2}$ $C_{12}^{2,3}$

Fig. 7

(3)The case $C_4 = C_4^3$. The patterns of 12-cycles contained in $Q_4 - C_4^3$ (Fig.8) are $C_{12}^{3,1}$, $C_{12}^{3,2}$, $C_{12}^{3,3}$ (Fig.9) (up to rotation and reversing of Q_4)

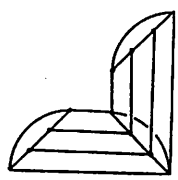


Fig. 8

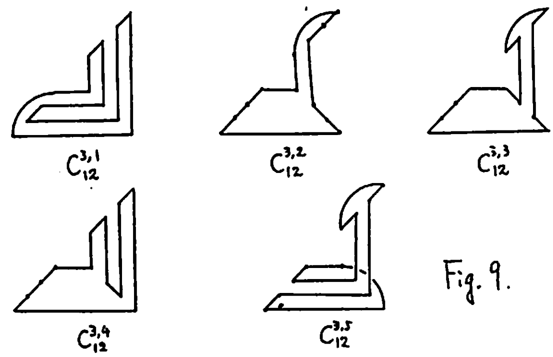


Fig. 9.

It is necessary $Q_4 - E(C_4^i \cup C_{12}^{i,j})$ (i, j as above) connected to be $C(\Delta) = C_{12} \cup C_4$. But all 11 cases of $Q_4 - E(C_4^i \cup C_{12}^{i,j})$ as above is not connected. Therefore it does not happen $C(\Delta) = C_{12} \cup C_4$. □

Proposition 4 *It does not happen the case* $C(\Delta) = \bigcup_{i=1}^4 C_4^i$.

Proof. We choose four 4-cycles from { C_4^1, C_4^2, C_4^3} (Fig. 3) in Q_4 repeatedly (repeated combination ${}_3H_4$) and dispose disjointly. There are 15 cases of choosing. Each case of 15 choosings of $Q_4 - E(\bigcup_{i=1}^4 C_4^i)$ can not happen or is not connected. So it does not happen the case $C(\Delta) = \bigcup_{i=1}^4 C_4^i$. □

Proposition 5 *It does not happen the case* $C(\Delta) = C_8^1 \cup C_8^2$.

Proof. The patterns of 8-cycle in Q_4 are C_8^i ($i = 1, 2, \cdots, 6$) as follows (Fig.10)

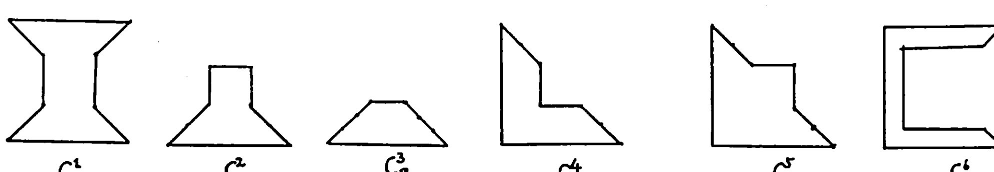


Fig. 10

We chooce two 8-cycles from C_8^i ($i = 1, 2, \cdots, 6$) and dispose disjointly. There are 21 cases of choosings (${}_6H_2 = 21$). Each case of 21 choosings can not happen or is not connected. So it does not happen the case $C(\Delta) = C_8^1 \cup C_8^2$. □
In the following section we will show that the remaining cases (1),(3),(5),(6) happen

and there are 5 classes of (1), 1 class of (3), 2 classes of (5) and 1 class of (6). So we obtain a following.

Theorem 1 $\#\{[Q_4, \Delta]\} \geq 9$.

4 Example

(I) Δ uses 4 floating edges of a square nested diagram of Q_4.

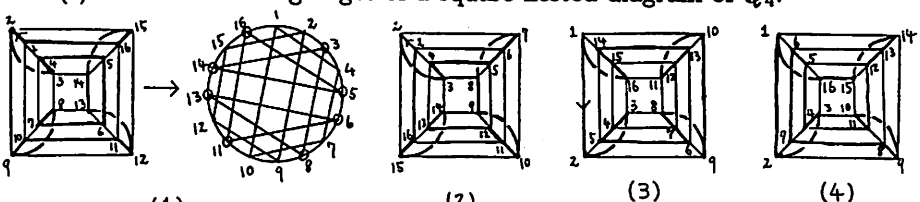


Fig. 11.

(1) $C(\Delta) = C_8 \cup C_4 \cup C_4$ and $\Delta \cap C_8$ has two pairs of adjacent vertices on Δ.
(2) $C(\Delta) = C_6 \cup C_6 \cup C_4$
(3) $C(\Delta) = C_8 \cup C_4 \cup C_4$ and $\Delta \cap C_8$ has four pairs of adjacent vertices on Δ.
(4) $C(\Delta) = C_{10} \cup C_6$

(II) Δ uses 3 floating edges of a square nested diagram of Q_4.

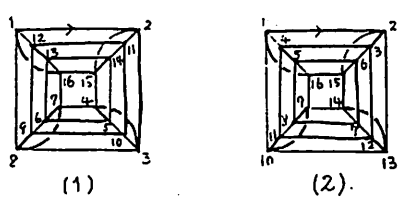


Fig. 12.

(1) $C(\Delta) = C_{16}$ (i.e. $C(\Delta) = C_{16}$ is also a H-cycle)

A sequence of vertices of C_{16} according to the order of vertices of Δ is
[1 8 3 10 5 14 11 2 15 4 7 16 13 6 9 12 1]

A sequence of length of the edges of C_{16} with respect to Δ is
[7 5 7 5 7 3 7 3 5 3 7 3 7 3 3 5]

A sequence of vertices of Δ according to the order of vertices of C_{16} is
[1 8 3 10 5 14 11 2 15 4 7 16 13 6 9 12 1]

A sequence of length of the edges of Δ with respect to C_{16} is
[7 5 7 5 7 3 7 3 5 3 7 3 7 3 3 5]

So there is an automorphism $\varphi \in Aut(Q_4)$ with $\varphi(\Delta) = C_{16}$

(2) $C(\Delta) = C_{16}$.

A sequence of vertices of C_{16} according to the order of vertices of Δ is
[1 4 11 8 5 16 9 14 7 12 3 6 15 2 13 10 1]

A sequence of length of the edges of C_{16} with respect to Δ is
[3 7 3 3 5 7 5 7 5 7 3 7 3 5 3 7]
A sequence of vertices of Δ according to the order of vertices of C_{16} is
[1 4 7 16 13 6 9 14 11 2 15 8 3 10 5 12 1]
A sequence of length of the edges of Δ with respect to C_{16} is
[3 3 7 3 7 3 5 3 7 3 7 5 7 5 7 5]
So there is an automorphism $\varphi \in Aut(Q_4)$ with $\varphi(\Delta) = C_{16}$

(III) Δ uses 2 floating edges of a square nested diagram of Q_4.

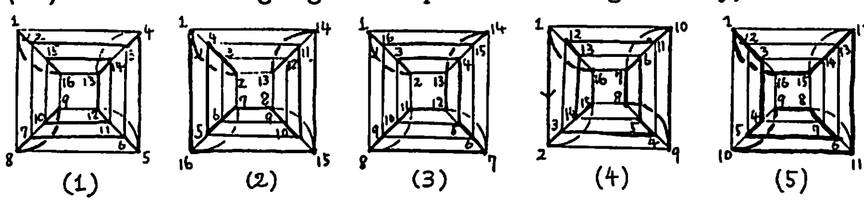


Fig. 13.

(1) $\mathcal{C}(\Delta) = C_8 \cup C_4 \cup C_4$ and $\Delta \cap C_8$ has four pairs of adjacent vertices on Δ.
(2) $\mathcal{C}(\Delta) = C_8 \cup C_4 \cup C_4$ and $\Delta \cap C_8$ has two pairs of adjacent vertices on Δ.
(3) $\mathcal{C}(\Delta) = C_{16}$.
A sequence of vertices of C_{16} according to the order of vertices of Δ is
[1 8 11 2 13 4 15 6 9 16 3 10 5 12 7 14 1]
A sequence of length of the edges of C_{16} with respect to Δ is
[7 3 7 5 7 5 7 3 7 3 7 5 7 5 7 3]
A sequence of vertices of Δ according to the order of vertices of C_{16} is
[1 8 11 2 13 4 15 6 9 16 3 10 5 12 7 4 1]
A sequence of length of the edges of Δ with respect to C_{16} is equal to that of C_{16} with respect to Δ.
So there is an automorphsm φ of Q_4 with $\varphi(\Delta) = C_{16}$
(4) $\mathcal{C}(\Delta) = C_{16}$.
A sequence of vertices of C_{16} according to the order of vertices of Δ is
[1 10 7 16 13 6 11 4 9 2 15 8 5 14 3 12 1]
A sequence of length of the edges of C_{16} with respect to Δ is
[7 3 7 3 7 5 7 5 7 3 7 3 7 5 7 5]
A sequence of vertices of Δ according to the order of vertices of C_{16} is
[1 10 15 8 13 6 3 12 9 2 7 16 5 14 11 4 1]
A sequence of length of the edges of Δ with respect to C_{16} is
[7 5 7 5 7 3 7 3 7 5 7 5 7 3 7 3].
We define an automorphism φ of Q_4 as follows;
$13 \longrightarrow 1,\ 6 \longrightarrow 10,\ 11 \longrightarrow 15,\ 4 \longrightarrow 8,\ 9 \longrightarrow 13,\ 2 \longrightarrow 6,\ 15 \longrightarrow 3,$

$8 \longrightarrow 12,\ 5 \longrightarrow 9,\ 14 \longrightarrow 2,\ 3 \longrightarrow 7,\ 12 \longrightarrow 16,\ 1 \longrightarrow 5,\ 10 \longrightarrow 14,$
$7 \longrightarrow 11,\ 16 \longrightarrow 4.$

Then $\varphi(\Delta) = C_{16}$.

(5) $\mathcal{C}(\Delta) = C_{10} \cup C_6$.

(IV) Δ uses 1 floating edges of a square nested diagram of Q_4.

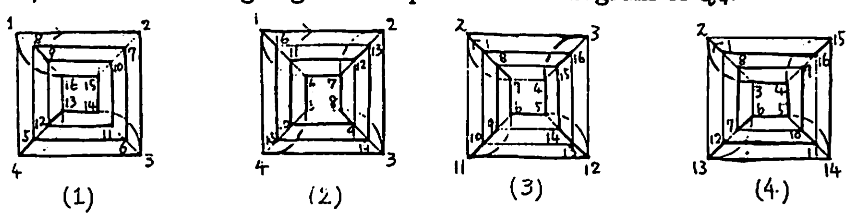


Fig. 14

(1) $\mathcal{C}(\Delta) = C_8 \cup C_4 \cup C_4$ and $\Delta \cap C_8$ has four pairs of adjacent vertices on Δ.

(2) $\mathcal{C}(\Delta) = C_{16}$.

A sequence of vertices of C_{16} according to the order of vertices of Δ is

[1 6 11 16 13 2 7 12 9 14 3 8 5 10 15 4 1]

A sequence of length of the edges of C_{16} with respect to Δ is

[5 5 5 3 5 5 5 3 5 5 5 3 5 5 5 3]

A sequence of length of the edges of Δ with respect to C_{16} is
the same as that of C_{16} with respect to *Delta*.

So there is an automorphism φ of Q_4 with $\varphi(\Delta) = C_{16}$.

(3) $\mathcal{C}(\Delta) = C_{16}$.

A sequence of vertices of C_{16} according to the order of vertices of Δ is

[1 8 15 4 7 2 11 6 9 14 5 12 3 16 13 10]

A sequence of length of the edges of C_{16} with respect to Δ is

[7 7 5 3 5 7 5 3 5 7 7 7 3 3 3 7]

A sequence of length of the edges of Δ with respect to C_{16} is
the same as that of C_{16}.

So there is an automorphism φ of Q_4 with $\varphi(\Delta) = C_{16}$.

(4) $\mathcal{C}(\Delta) = C_{16}$.

A sequence of vertices of C_{16} according to the order of vertices of Δ is

[1 8 3 6 13 2 15 4 9 16 11 14 5 10 7 2]

A sequence of length of the edges of C_{16} with respect to Δ is

[7 5 3 7 5 3 5 5 7 5 3 7 5 3 5 5]

And a sequence of length of edges of Δ with respect to C_{16} is the same as that of C_{16}.

So there is an automorphism φ of Q_4 with $\varphi(\Delta) = C_{16}$.

(V) Δ does not uses floating edge of a square nested diagram of Q_4.

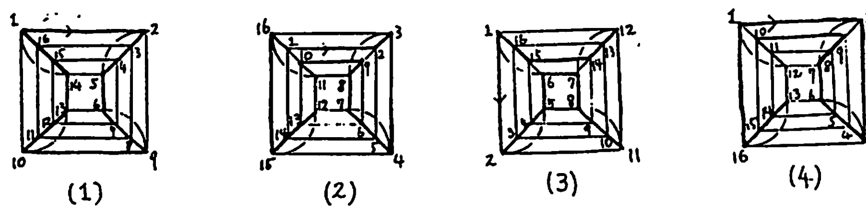


Fig. 15.

(1) $\mathcal{C}(\Delta) = C_8 \cup C_4 \cup C_4$ and $\Delta \cap C_8$ has four pairs of adjacent vertices on Δ.
(2) $\mathcal{C}(\Delta) = C_8 \cup C_4 \cup C_4$ and $\Delta \cap C_8$ has four pairs of adjacent vertices on Δ.
(3) $\mathcal{C}(\Delta) = C_8 \cup C_4 \cup C_4$ and $\Delta \cap C_8$ has two pairs of adjacent vertices on Δ.
(4) $\mathcal{C}(\Delta) = C_8 \cup C_4 \cup C_4$ and $\Delta \cap C_8$ has two pairs of adjacent vertices on Δ.

For the following example in I$\sim$V, we can make an automorphism φ sending each other as follows.

$(I-1) \longrightarrow (III-2)$ by $\varphi(i) = i+2$ where $\varphi(15) = 1, \varphi(16) = 2$.
$(I-3) \longrightarrow (V-1)$ by $\varphi(i) = i$ $(i = 1, 2, \cdots, 16)$.
$(III-1) \longrightarrow (IV-1)$ by $\varphi(i) = i$ $(i = 1, 2, \cdots, 16)$.
$(III-5) \longrightarrow (I-4)$ by $\varphi(i) = i+4$ where $\varphi(13) = 1$ etc.
$(IV-1) \longrightarrow (V-1)$ by $\varphi(i) = i+1$ where $\varphi(16) = 1$.
$(V-1) \longrightarrow (V-2)$ by $\varphi(i) = i+1$ where $\varphi(16) = 1$.
$(V-3) \longrightarrow (I-1)$ by $\varphi(i) = i+2$ where $\varphi(15) = 1, \varphi(16) = 2$.
$(V-4) \longrightarrow (V-3)$ by $\varphi(i) = i+2$ where $\varphi(15) = 1, \varphi(16) = 2$.

Remark. E.N.Gilbert already proved $\#\{[Q_4, \Delta]\} = 9$ ([G]). But we could not understand the structure of the complement of the Hamilton cycle from that paper. So we studied $\#\{[Q_4, \Delta]\}$ from the other view point.

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