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3-球面の特徴付け II

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箱根セミナー 1999 では 3-球面 の特徴付を示し、その特徴付をもとにポアンカレ予想へのアプローチを試みました。その後の検討の結果、まだ証明が不十分であるようです。

この報告は、「箱根セミナー 1999」で示した 3-球面 の特徴付の改良版です。すなわち、より緩やかな条件のもとで多様体が 3-球面であることを示します。以下論文の原稿で、英文になることをお許し下さい。

Introduction

A *Heegaard splitting* (or an *H-splitting* for short) $\mathcal{H} = (H_1, H_2)$ of a closed orientable 3-manifold M is a decomposition as a union of two handle bodies $M = H_1 \cup H_2$ such that $H_1 \cap H_2 = \partial H_1 = \partial H_2$. Given a closed orientable 3-manifold M and its H-splitting $\mathcal{H} = (H_1, H_2)$ with genus $g = g(\mathcal{H})$, we have shown in [1] that the localness of some $(g-d)$ component link in H_1, so called a *d-pseudo core* (cf. the Definition 0.1 below), gives an estimation $HG(M) \leq d$, where $HG(M)$ is the Heegaard genus, namely the minimum genus among all the H-splittings of M.

Definition 0.1. *A link* $\mathcal{L} = (\alpha_1, \ldots, \alpha_{g-d})$ *in* H_1 *with* $(g-d)$ *components* ($g = g(\mathcal{H})$, $d = 0, 1, \ldots, g-1$) *is said to be a* d-pseudo core *of* $\mathcal{H}$ *if there is a family* $\vec{D} = (D_1, D_2, \ldots, D_{g-d})$ *(possibly* $D_i \cap D_j \neq \emptyset$*) of meridian disks of* H_1 *which satisfies the following two conditions (C.1) and (C.2):*

(C.1) $D_p \cap \alpha_p$ *consists of exactly one point for any* $p = 1, 2, \ldots, g-d$,

(C.2) $D_q \cap \alpha_p = \emptyset$ *for* $q > p$.

We call such a family $\vec{D}$ *an associated meridian disk family.*

The result in [1] is the following :

Theorem 0.1. *If a closed orientable 3-manifold M admits an H-splitting with a local d-pseudo core, then $HG(M) \leq d$,* where "local" means "completely included in a 3-ball in M".

As an immediate consequence, we have that

Corollary 0.2. *If a closed orientable 3-manifold M admits an H-splitting with a local 0-pseudo core, then M is homeomorphic to a standard 3-sphere S^3.*

In the case of a homotopy 3-sphere M (see [2]), we can find an H-splitting $\mathcal{H} = (H_1, H_2)$ of M with a 0-pseudo core $\mathcal{L} \subset H_1$ satisfying that

(i) each component of $\mathcal{L}$ is local in M, but

(ii) $\mathcal{L}$ itself is not necessarily local (*i.e.* we cannot conclude $\mathcal{L}$ to be local).

In this paper, given an H-splitting $\mathcal{H}$ having a 0-pseudo core with the above properties (i) and (ii), we seek a condition for getting another H-splitting $\mathcal{H}'$, called an *induced H-splitting* (see §2), which admits a *local* 0-pseudo core.

1. Definitions and Statement of the Result

1.1. Localizing arc system for a link. Let $L = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a link in M, and let θ be an oriented arc going from a point on a component λ_i to another point on a component λ_j. A *cross change along θ* is an operation of deforming λ_i into a band sum of λ_i with a meridian circle of λ_j. The result of this deformation is denoted by $L[\theta]$. Of course $L[\theta]$ is determined up to twisting of the band centered at the arc θ. For a family $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$ of disjoint oriented arcs θ_j each of which connects two points on K, the link obtained by successive cross changes along $\theta_1, \theta_2, \ldots, \theta_m$ is denoted by $L[\vec{\theta}]$.

Definition 1.1. *A family $\vec{\theta} = (\theta_1, \ldots, \theta_m)$ of oriented arcs as above is said to be a* localizing arc system *for the link L if the cross changed link $L[\vec{\theta}]$ is local for a suitable choice of twisting of bands centered at θ_k $(k = 1, \ldots, m)$. (See* [1] *for details.)*

The following lemma gives a basic condition under which we can deform a d-pseudo core, which is not necessarily local, to a local one.

Lemma 1.1. *Let* $\mathcal{H} = (H_1, H_2)$ *be an H-splitting of* M *with genus* g, *and* $\mathcal{L} = (\alpha_1, \alpha_2, \ldots, \alpha_{g-d}) \subset H_1$ *be a d-pseudo core (not necessarily local) with an associated meridian disk family* $\vec{D} = (D_1, D_2, \ldots, D_{g-d})$. *If* $\mathcal{L}$ *has a localizing arc system* $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ *satisfying that*

(*) $D_{p'} \cap \theta_i = \emptyset$ *whenever the i-th oriented arc* θ_i *starts at* α_p *and* $p' \geq p$,

then $\mathcal{H}$ *admits a local d-pseudo core, and so* $HG(M) \leq d$.

Proof. The definition of a d-pseudo core and the condition (*) imply that the link $\mathcal{L}[\vec{\theta}]$, the link obtained from $\mathcal{L}$ by cross changes along the oriented arcs θ_j, is also a d-pseudo core having the same associated meridian disk family, that is, the H-splitting $\mathcal{H}$ admits a local d-pseudo core $\mathcal{L}[\vec{\theta}]$. □

In the case where the deficiency $d = 0$, this lemma itself gives a sufficient condition for M to be homeomorphic to a standard 3-sphere S^3.

1.2. A simple version of the main results. Because conditions in our main result, which is stated in the next subsection, is too much complicated to read off its meanings, we first state a rather simple version of it in this subsection.

Let $\mathcal{H} = (H_1, H_2)$ be an H-splitting of M with genus g, and $\vec{D}_1 = (D_{1,1}, D_{1,2}, \ldots, D_{1,g})$ and $\vec{D}_2 = (D_{2,1}, D_{2,2}, \ldots, D_{2,g})$ be complete systems of meridian disks of H_1 and H_2 respectively. A *Heegaard diagram* induced by a pair $(\vec{D}_1, \vec{D}_2)$ is a triple $(F; \vec{y}, \vec{z})$; $F = \partial H_1 = \partial H_2$ is the Heegaard surface, and $\vec{y} = (y_1, y_2, \ldots, y_g)$ and $\vec{z} = (z_1, z_2, \ldots, z_g)$ are loops on F bounding $\vec{D}_1$ and $\vec{D}_2$, namely $y_j = \partial D_{1,j}$ and $z_j = \partial D_{2,j}$ $(j = 1, 2, \ldots, g)$. As usual, we assume that the intersection $\vec{y} \cap \vec{z}$ consists of finitely many points. The definition of a simple version of a *reducing pair* is given as follows.

Definition 1.2. *A pair* $(\vec{D}_1, \vec{D}_2)$ *of families of* g *meridian disks of* H_1 *and of* H_2 *is said to be a* simple reducing pair *if it and its induced triple* $(F; \vec{y}, \vec{z})$ *satisfy the following four conditions (A.1)–(A.4).*

(A.1) *For the* μ_0 *disks* $D_{1,p}$ $(1 \leq p \leq \mu_0)$ *in the system* $\vec{D}_1$, *there are* μ_0 *loops* $\alpha_{1,p}$ $(p = 1, 2, \ldots, \mu_0)$ *in* H_1 *such that*

- (i) $D_{1,p} \cap \alpha_{1,p}$ *consists of exactly one point, and*
- (ii) $D_{1,p'} \cap \alpha_{1,p} = \emptyset$ *for any* p' *with* $p + 1 \leq p' \leq g$.

(A.2) *For the* ν_0 *disks* $D_{2,q}$ $(1 \le q \le \nu_0)$ *in the system* $\vec{D}_2$, *there are* ν_0 *loops* $\alpha_{2,q}$ $(q = 1, 2, \ldots, \nu_0)$ *in* H_2 *such that*

(i) $D_{2,q} \cap \alpha_{2,q}$ *consists of exactly one point, and*

(ii) $D_{2,q'} \cap \alpha_{2,q} = \emptyset$ *for any* q' *with* $q+1 \le q' \le g$.

(A.3) *The* $(2g-\mu_0-\nu_0)$ *loops* y_j $(\mu_0+1 \le j \le g)$ *and* z_j $(\nu_0+1 \le j \le g)$ *do not divide the surface* F, *that is,* $F - ((\bigcup_{p=\mu_0+1}^{g} y_p) \cup (\bigcup_{q=\nu_0+1}^{g} z_q))$ *is connected.*

(A.4) *The link* $\mathcal{L}$ *which consists of the* $(\mu_0 + \nu_0)$ *components* $\alpha_{1,p}$ $(1 \le p \le \mu_0)$ *and* $\alpha_{2,q}$ $(1 \le q \le \nu_0)$ *has a loclizing arc system* $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ *such that*

(i) *there is a component* X_0 *of* $F - (\vec{y} \cup \vec{z})$ *such that* $\vec{\theta} \cap F \subset X_0$,

(ii) *if* θ_i *starts at* $\alpha_{1,p}$ *for some* p *with* $1 \le p \le \mu_0$, *then* $\theta_i \cap \vec{D}_2 = \emptyset$, *and* $\theta_i \cap D_{1,p'} = \emptyset$ *for any* p' *with* $p \le p' \le g$,

(iii) *if* θ_i *starts at* $\alpha_{2,q}$ *for some* q *with* $1 \le q \le \nu_0$, *then* $\theta_i \cap D_{1,p} = \emptyset$ *for any* p *with* $\mu_0 + 1 \le p \le g$, *and* $\theta_i \cap D_{1,q'} = \emptyset$ *for any* q' *with* $q \le q' \le g$.

The following theorem will be proved in the next section.

Theorem 1.2. *An orientable closed manifold* M *is homeomorphic to a standard 3-sphere* S^3 *if* M *has an H-splitting* $\mathcal{H} = (H_1, H_2)$ *which admits a simple reducing pair* $(\vec{D}_1, \vec{D}_2)$ *of complete meridian disk systems of* H_1 *and* H_2.

Remark 1. The condition $(*)$ in Lemma 1.1 for $d = 0$ is a special case of the above conditions (A.1)–(A.4) where $\mu_0 = g$ and $\nu_0 = 0$. The above theorem implies that we might confirm M to be a 3-sphere S^3 even if we cannot find a local 0-pseudo core in a handle body H_1. Hence this theorem gives a weaker condition for M to be S^3 than Corollay 0.2.

An easy corollary of this theorem is given in the following.

Corollary 1.3. *If an orientable closed 3-manifold* M *admits a H-splitting having a induced triple* $(F; \vec{y}, \vec{z})$ *such that* $F-(\vec{y} \cup \vec{z})$ *is connected, then* M *is homeomorphic to a standard 3-sphere* S^3.

Indeed, putting $\mu_0 = \nu_0 = 0$, we can see that such a Heegaard diagram guarantees the conditions (A.1)–(A.4) for a simple reducing pair. Hence Theorem 1.2 immediately implies the corollary.

1.3. Statement of the main result. The definition of a *reducing pair* of complete meridian disk systems is given as follows, which is a generalization of the above defined simple reducing pair.

Definition 1.3. *A pair* $(\vec{D}_1, \vec{D}_2)$ *of complete meridian disk systems of* H_1 *and of* H_2 *is said to be a* reducing pair *if, for the pair* $(\vec{D}_1, \vec{D}_2)$ *and its induced triple* $(F; \vec{y}, \vec{z})$, *there are a component* X_0 *of* $F-(\vec{y}\cup\vec{z})$ *and two non-decreasing sequences of non negative integers*

$$0 \equiv \mu'_0 \le \mu_0 \le \mu'_1 \le \mu_1 \le \mu'_2 \le \mu_2 \le \cdots\cdots \le \mu'_N \le \mu_N \le g$$
$$0 \equiv \nu'_0 \le \nu_0 \le \nu'_1 \le \nu_1 \le \nu'_2 \le \nu_2 \le \cdots\cdots \le \nu'_N \le \nu_N \le g$$

which satisfy the following $(4N+4)$ *conditions (B.1)*$_\ell$, *(B.2)*$_\ell$ *(for* $0 \le \ell \le N$*), (B.4)*$_\ell$, *(B.4)*$_\ell$ *(for* $1 \le \ell \le N$*), (B.5) and (B.6).*

(B.1)$_\ell$ $(\ell = 0, 1, 2, \ldots, N)$

For p *with* $\mu'_\ell + 1 \le p \le \mu_\ell$, *there are loops* $\alpha_{1,p}$ $(\mu'_\ell + 1 \le p \le \mu_\ell)$ *in the interior of* H_1 *such that*

(i) $D_{1,p} \cap \alpha_{1,p}$ *consists of exactly one point, and*

(ii) $D_{1,p'} \cap \alpha_{1,p} = \emptyset$ *for any* p' *with* $p+1 \le p' \le g$,

(B.2)$_\ell$ $(\ell = 0, 1, 2, \ldots, N)$

For q *with* $\nu'_\ell + 1 \le q \le \nu_\ell$, *there are loops* $\alpha_{2,q}$ $(\nu'_\ell + 1 \le q \le \nu_\ell)$ *in the interior of* H_2 *such that*

(i) $D_{2,q} \cap \alpha_{2,q}$ *consists of exactly one point, and*

(ii) $D_{2,q'} \cap \alpha_{2,q} = \emptyset$ *for any* q' *with* $q+1 \le q' \le g$,

(B.3)$_\ell$ $(\ell = 1, 2, \ldots, N)$

For p *with* $\mu_{\ell-1} + 1 \le p \le \mu'_\ell$, *there are loops* $\tilde{\alpha}_{1,p}$ $(\mu_{\ell-1} + 1 \le p \le \mu'_\ell)$ *on the surface* F *such that*

(i) $\tilde{\alpha}_{1,p}$ *is included in the same component of*

$$F - (\{y_j \,|\, p+1 \le j \le g\} \cup \{z_j \,|\, \nu_{\ell-1} + 1 \le j \le g\})$$

as the specified component X_0,

(ii) $\widetilde{\alpha}_{1,p}$ *intersects* y_p *at exactly one point.*

$(\mathrm{B}.4)_\ell$ $(\ell = 1, 2, \ldots, N)$

For q *with* $\nu_{\ell-1}+1 \le q \le \nu'_\ell$, *there are loops* $\widetilde{\alpha}_{2,q}$ $(\nu_{\ell-1}+1 \le q \le \nu'_\ell)$ *on the surface* F *such that*

(i) $\widetilde{\alpha}_{2,q}$ *is included in the same component of*

$$F - (\{y_j \,|\, \mu'_\ell + 1 \le j \le g\} \cup \{z_j \,|\, q+1 \le j \le g\})$$

as the specified component X_0,

(ii) $\widetilde{\alpha}_{2,q}$ *intersects* z_q *at exactly one point.*

(B.5) $F - (\{y_j \,|\, \mu_N + 1 \le j \le g\} \cup \{z_j \,|\, \nu_N + 1 \le j \le g\})$ *is connected.*

(B.6) *The link* $\mathcal{L}$ *which consists of the* $n_* \equiv \sum_{\ell=0}^{N}((\mu_\ell - \mu'_\ell) + (\nu_\ell - \nu'_\ell))$ *components* $\alpha_{1,p}$ $(\mu'_\ell + 1 \le p \le \mu_\ell,\ 0 \le \ell \le N)$ *and* $\alpha_{2,q}$ $(\nu'_\ell + 1 \le q \le \nu_\ell,\ 0 \le \ell \le N)$ *is local in* M.

Remark 2. The last condition (B.6) can be weakened into the following (B.6)$'$:

(B.6)$'$ The link $\mathcal{L}$ which consists of the $n_* \equiv \sum_{\ell=0}^{N}((\mu_\ell - \mu'_\ell) + (\nu_\ell - \nu'_\ell))$ components $\alpha_{1,p}$ $(\mu'_\ell + 1 \le p \le \mu_\ell,\ 0 \le \ell \le N)$ and $\alpha_{2,q}$ $(\nu'_\ell + 1 \le q \le \nu_\ell,\ 0 \le \ell \le N)$ has a loclizing arc system $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ such that

(i) if θ_i starts at $\alpha_{1,p}$ for some p with $\mu'_{\ell_i} + 1 \le p \le \mu_{\ell_i}$, then

$$F \cap \theta_i \subset \widetilde{X}_{\ell_i}$$
$$\theta_i \cap D_{1,p'} = \emptyset \quad \text{for any} \quad p' \quad \text{with} \quad p \le p' \le g, \text{ and}$$
$$\theta_i \cap D_{2,q} = \emptyset \quad \text{for any} \quad q \quad \text{with} \quad \nu'_{\ell_i} + 1 \le q \le g,$$

(ii) if θ_i starts at $\alpha_{2,q}$ for some q with $\nu'_{\ell_i} + 1 \le q \le \nu_{\ell_i}$, then

$$F \cap \theta_i \subset \widetilde{X}_{\ell_i}$$
$$\theta_i \cap D_{1,p} = \emptyset \quad \text{for any} \quad p \quad \text{with} \quad \mu_{\ell_i} + 1 \le p \le g, \text{ and}$$
$$\theta_i \cap D_{2,q'} = \emptyset \quad \text{for any} \quad q' \quad \text{with} \quad q \le q' \le g,$$

where $\widetilde{X}_\ell$ denotes the union of components X of $F-(\vec{y}\cup\vec{z})$ such that there is a number j with $1\le j\le \ell$ for which $X\cap\tilde{\alpha}_{1,p}\neq\emptyset$ for some $\mu_{j-1}\le p\le\mu'_j$ or $X\cap\tilde{\alpha}_{2,q}\neq\emptyset$ for some $\nu_{j-1}\le q\le\nu'_j$.

Remark 3. In the weakened form, a reducing pair is a simple reducing pair if $N=0$.

Our main result in this paper is the following theorem, which is proved in §3.

Theorem 1.4. *An orientable closed manifold M is homeomorphic to a standard 3-sphere S^3 if M has an H-splitting $\mathcal{H}=(H_1,H_2)$ which admits a reducing pair $(\vec{D}_1,\vec{D}_2)$ of complete meridian disk systems of H_1 and H_2.*

2. Proof of Theorem 1.2

Let $\mathcal{H}=(H_1,H_2)$ be an H-splitting of M which admits a reducing pair $(\vec{D}_1,\vec{D}_2)$ of complete meridian disk systems, and $(F;\vec{y},\vec{z})$ be the H-diagram induced by $\vec{D}_1$ and $\vec{D}_2$. Taking an adequate family of arcs $\vec{\xi}=(\xi_1,\xi_2,\dots,\xi_m)$ on F, we can make each component of $F-(\vec{y}\cup\vec{z}\cup\vec{\xi})$ be homeomorphic to an open 2-disk, where we may assume that each ξ_j is an arc connecting two points on $\vec{y}\cup\vec{z}$.

First we shall prove that

Lemma 2.1. *The decomposition $\mathcal{H}'=(H'_1,H'_2)$, where H'_1 is the exterior of the 1-complex $\vec{y}\cup\vec{z}\cup\vec{\xi}$ and H'_2 is its regular neighborhood, gives an H-splitting of M.*

Proof. Let $D_{1,j}$ (respectively $D_{2,j}$) be the meridian disk in the complete system $\vec{D}_1$ (respectively $\vec{D}_2$) with $\partial D_{1,j}=y_j$ (respectively $\partial D_{2,j}=z_j$). Then the H-diagram and the family $\vec{\xi}$ of arcs on F give a structure $\mathcal{K}=\{\mathcal{K}_q\,|\,q=0,1,2,3\}$ of a cell complex, that is,

(0) the set $\mathcal{K}_0$ of 0-cells consists of the points in $\vec{y}\cap\vec{z}$ and the end points of ξ_j,
(1) the set $\mathcal{K}_1$ of 1-cells consists of the arcs ξ_j and the components of $(\vec{y}\cup\vec{z})-\mathcal{K}_0$,
(2) the set $\mathcal{K}_2$ of 2-cells consists of the meridian disks in the systems $\vec{D}_j$ ($j=1,2$) and the components of $F-(\vec{y}\cup\vec{z}\cup\vec{\xi})$, and
(3) the set $\mathcal{K}_3$ of 3-cells consists of two 3-balls $H_1-\vec{D}_1$ and $H_2-\vec{D}_2$.

Obviously H'_2 is a handle body. On the other hand, the complementary set H'_1 can be thought to be a regular neighborhood of the 1-skeleton of the cell complex $\mathcal{K}^*$ which is dual to $\mathcal{K}$. Hence H'_1 is also a handle body. □

We call the above H-splitting $\mathcal{H}'$ an *induced H-plitting* for a pair $(\vec{D}_1, \vec{D}_2)$ of meridian disk systems. Assume that $F-(\vec{y}\cup\vec{z}\cup\vec{\xi})$ has (m_0+1) components which we denote by $Y_0, Y_1, \ldots, Y_{m_0}$, and assume $Y_0 \subset X_0$, where X_0 is the component of $F-(\vec{y}\cup\vec{z})$ whose existence is assumed in the condition (A.4).

Lemma 2.2. *The above defined handle body H_1' has a complete meridian disk system, which we denote by $\vec{D}'$, consisting of $g' \equiv 2g+m_0$ disks*

$$D_{\nu,j}\cap H_1' \quad (\nu=1,2,\ j=1,2,\ldots,g), \qquad Y_i\cap H_1' \quad (i=1,2,\ldots,m_0),$$

and so the genus of H_1' is g'.

Remark 4. It is to be noticed that $Y_0\cap H_1'$ is *not* a member of the above defined complete meridian disk system.

Proof. Since $H_1-\vec{D}_1$ and $H_2-\vec{D}_2$ are both open 3-balls and Y_0 is an open 2-disk, $W\equiv(M-(F\cup\vec{D}_1\cup\vec{D}_2))\cup Y_0$ is homeomorphic to an open 3-ball. Hence the interior of $H_1'-\vec{D}'$, which is ambient isotopic to W, is also an open 3-ball. This implies that $\vec{D}'$ is a complete meridian disk system. □

For the proof of the theorem, it is sufficient to show that, giving an adequate ordering to the disks in $\vec{D}'$, we can make $\vec{D}'$ a system associated with a local pseudo core. The ordering for the disks in $\vec{D}'$ is fixed as follows.

- For $1\le i\le \mu_0$,
  $D_i' = D_{1,p}\cap H_1'$ $(i=p,\ 1\le p\le \mu_0)$.
- For $\mu_0+1\le i\le \mu_0+\nu_0$,
  $D_i' = D_{2,q}\cap H_1'$ $(i=\mu_0+q,\ 1\le q\le \nu_0)$.
- For $\mu_0+\nu_0+1\le i\le \mu_0+\nu_0+m_0$,
  $D_i' = Y_j\cap H_1'$ $(i=\mu_0+\nu_0+j,\ 1\le j\le m_0)$.
- For $\mu_0+\nu_0+m_0+1\le i\le g+\nu_0+m_0$,
  $D_i' = D_{1,p}\cap H_1'$ $(i=\mu_0+\nu_0+m_0+(p-\mu_0),\ \mu_0+1\le p\le g)$.
- For $g+\nu_0+m_0+1\le i\le g'\equiv 2g+m_0$,
  $D_i' = D_{2,q}\cap H_1'$ $(i=g+\nu_0+m_0+(q-\nu_0),\ \nu_0+1\le q\le g)$.

For such an ordering for the meridian disks in the complete system $\vec{D}'$, we have that

Lemma 2.3. *There exists an ordered link* $\mathcal{L}' = (\alpha'_1, \alpha'_2, \dots, \alpha'_{g'})$ *in* H'_1 *such that*

(i) $\mathcal{L}'$ *is a* 0*-pseudo core which has* $\vec{D}'$ *as its associated meridian disk family,*

(ii) $\mathcal{L} \subset \mathcal{L}'$ *and* $\vec{\theta}$ *is a localizing arc system for* $\mathcal{L}'$, *where* $\mathcal{L}$ *and* $\vec{\theta}$ *are those in the condition* (A.4).

Proof. The condition (A.3) implies that, for adequately ordered components Y_j $(1 \le j \le m_0)$ of $F - (\vec{y} \cup \vec{z} \cup \vec{\xi})$, there is a sequence $w_1, w_2, \dots, w_{m_0}$ of 1-cells in the above mentioned cell complex $\mathcal{K}$ satisfying the following conditions (i) and (ii) :

(i) if $w_j \subset \vec{y} \cup \vec{z}$, then $w_j \subset \partial D_{1,p}$ for some p with $1 \le p \le \mu_0$ or $w_j \subset \partial D_{2,q}$ for some q with $1 \le q \le \nu_0$,

(ii) each w_j is included in $\partial Y_j \cap \partial Y_{j'}$ for some j' with $0 \le j' < j$.

To give the definition of loops α'_i, we introduce a loop $\varepsilon(\sigma)$ in H'_1 for each 1-cell σ in the cell complex $\mathcal{K}$ as follows; choose a small 2-disk $\Delta(\sigma)$ so that it is disjoint from $\vec{\theta}$, and $\Delta(\sigma) \cap (\vec{y} \cup \vec{z} \cup \vec{\xi})$ consists of exactly one point which is on σ, and define $\varepsilon(\sigma)$ to be $\varepsilon(\sigma) = \partial\Delta(\sigma)$. Notice that the intersections of such a loop $\varepsilon(\sigma)$ with 2-cells in $\mathcal{K}$ may be assumed as follows :

(i) if $\sigma \subset \vec{y} \cup \vec{z}$, then $\varepsilon(\sigma) \cap (F \cup \vec{D}_1 \cup \vec{D}_2)$ consists of three points, one of them is on $\vec{D}_1 \cup \vec{D}_2$ and the other two points are on F, and

(ii) if $\sigma \subset \vec{\xi}$, then $\varepsilon(\sigma) \cap (F \cup \vec{D}_1 \cup \vec{D}_2)$ consists of two points, both of which are on F.

Namely $\varepsilon(\sigma) \cap \vec{D}'$ consists of at most three points.

Now, corresponding to the meridian disks $D'_1, D'_2, \dots, D'_{g'}$ ordered as the above, define an ordered link $\mathcal{L}' = (\alpha'_1, \alpha'_2, \dots, \alpha'_{g'})$ in H'_1 as follows :

- For $1 \le i \le \mu_0$,
  $\alpha'_i = \alpha_{1,p}$ $(i = p,\ 1 \le p \le \mu_0)$,
- For $\mu_0 + 1 \le i \le \mu_0 + \nu_0$,
  $\alpha'_i = \alpha_{2,q}$ $(i = \mu_0 + q,\ 1 \le q \le \nu_0)$,
- For $\mu_0 + \nu_0 + 1 \le i \le \mu_0 + \nu_0 + m_0$,
  $\alpha'_i = \varepsilon(w_j)$ $(i = \mu_0 + \nu_0 + j,\ 1 \le j \le m_0)$,
- For $\mu_0 + \nu_0 + m_0 + 1 \le i \le g + \nu_0 + m_0$,

$\alpha'_i = \varepsilon(u_p)$ $(i = \mu_0 + \nu_0 + m_0 + (p - \mu_0),\ \mu_0 + 1 \le p \le g)$, where u_p is a 1-cell such that $u_p \subset y_p$,

- For $g + \nu_0 + m_0 + 1 \le i \le g' \equiv 2g + m_0$,
  $\alpha'_i = \varepsilon(v_q) \cap H'_1$ $(i = g + \nu_0 + m_0 + (q - \nu_0),\ \nu_0 + 1 \le q \le g)$, where v_q is a 1-cell such that $v_q \subset z_q$.

It is evident that $\mathcal{L}'$ is a link in H'_1 containing $\mathcal{L}$ as its sublink. It is also easy to see that the conditions (A.1)–(A.3) and the choice of 1-cells w_j, u_p and v_p imply that $\mathcal{L}'$ is a 0-pseudo core with an associated meridian disk family $\vec{D}'$.

The link $\mathcal{L}[\vec{\theta}]$, which is obtained from $\mathcal{L}$ by cross changes along $\vec{\theta}$, is local by the condition (A.4), and the components in $\mathcal{L}' - \mathcal{L}$, each of which is of the type $\varepsilon(\sigma)$ for some 1-cell σ, bound 2-disks which are mutually disjoint and do not intersect with $\vec{\theta}$. Hence the link $\mathcal{L}'[\vec{\theta}] = \mathcal{L}[\vec{\theta}] \cup (\mathcal{L}' - \mathcal{L})$ is also a local link. This completes the proof of Lemma 2.3. □

From the ordering for the components of $\mathcal{L}'$ and the conditions (A.1)–(A.4) it follows that $\mathcal{H}'$, $\mathcal{L}'$ and $\vec{\theta}$ satisfy the condition (∗) in Lemma 1.1 (for $d = 0$). Therefore Lemma 1.1 lead us to the conclusion that the H-splitting $\mathcal{H}'$ admits a local 0-pseudo core $\mathcal{L}'[\vec{\theta}]$, and so M is homeomorphic to S^3 by Corollary 0.2. This completes the proof of Theorem 1.2. □

3. Proof of Theorem 1.4

We will prove the main theorem in the weakened form, that is, we employ the condition (B.6)′ instead of (B.6). The proof of the main theorem is somewhat complicated, but it is done along just the same line as Theorem 1.2. We use the same induced H-splitting $\mathcal{H}'$ and the same notation as in the previous section. However the ordering for the meridian disks D'_j $(1 \le j \le g')$ will be changed adequately.

The condition $(B.3)_\ell$ implies that,for each p with $\mu_{\ell-1} \le p \le \mu'_\ell$, we can find a 1-cell $w_{1,p}$ in $\mathcal{K}$ on which there is the unique point $\tilde{\alpha}_{1,p} \cap D_{1,p}$. And the condition $(B.4)_\ell$ implies that,for each q with $\nu_{\ell-1} \le q \le \nu'_\ell$, we can take a 1-cell $w_{2,q}$ in $\mathcal{K}$ on which there is the unique point $\tilde{\alpha}_{2,q} \cap D_{2,q}$. Moreover those conditions imply that, for each p $(\mu_{\ell-1} \le p \le \mu'_\ell)$ and each q $(\nu_{\ell-1} \le q \le \nu'_\ell)$, there are sequences $u^1_{1,p}, u^2_{1,p}, \ldots, u^{s_p}_{1,p}$ and $u^1_{2,q}, u^2_{2,q}, \ldots, u^{t_q}_{2,q}$ of 1-cells in $\mathcal{K}$, and corresponding sequences $Z^1_{1,p}, Z^2_{1,p}, \ldots, Z^{s_p}_{1,p}$ and $Z^1_{2,q}, Z^2_{2,q}, \ldots, Z^{t_q}_{2,q}$ of 2-cells in $\mathcal{K}$ which satisfy the following conditions (i)–(vii) :

(i) $Z^k_{i,r} = Y_j$ for some $1 \le j \le m_0$, and $Z^k_{i,r} = Z^{k'}_{i',k'}$ if and only if $(k,i,r) = (k',i',r')$,

(ii) the loop $\tilde{\alpha}_{1,p}$ in the condition $(B.3)_\ell$ is included in $\tilde{Z}_{1,p}$, where $\tilde{Z}_{1,p}$ is the closure of the set

$$Y_0 \cup (\bigcup\{Z^k_{1,p'} \,|\, p' \le p,\ 1 \le k \le s_{p'}\}) \cup (\bigcup\{Z^k_{2,q} \,|\, q \le p-1,\ 1 \le k \le t_q\}) ,$$

(iii) if $u^k_{1,p} \subset \vec{y} \cup \vec{z}$, then $u^k_{1,p} \subset \partial D_{1,p'}$ for some p' with $1 \le p' \le p-1$ or $u^k_{1,p} \subset \partial D_{2,q}$ for some q with $1 \le q \le \nu_{\ell-1}$,

(iv) $u^k_{1,p} \subset \partial Z^k_{1,p} \cap \tilde{Z}_{1,p-1} \cup \tilde{Z}_{2,p-1} \cup (\bigcup_{k'=1}^{k-1} \partial Z^{k'}_{1,p})$, where $\tilde{Z}_{2,q}$ denotes the closure of the set

$$Y_0 \cup (\bigcup(\{Z^k_{1,p'} \,|\, p' \le q,\ 1 \le k \le s_{p'}\}) \cup (\bigcup\{Z^k_{2,q'} \,|\, q' \le q,\ 1 \le k \le t_{q'}\}) ,$$

(v) the loop $\tilde{\alpha}_{2,q}$ in the condition $(B.4)_\ell$ is included in $\tilde{Z}_{2,q}$,

(vi) if $u^k_{2,q} \subset \vec{y} \cup \vec{z}$, then $u^k_{2,q} \subset \partial D_{1,p'}$ for some p' with $1 \le p' \le \mu'_\ell$ or $u^k_{2,q} \subset \partial D_{2,q'}$ for some q' with $1 \le q' \le q-1$,

(vii) $u^k_{2,q} \subset \partial Z^k_{2,q} \cap (\tilde{Z}_{1,p} \cup \tilde{Z}_{2,q-1} \cup (\bigcup_{k'=1}^{k-1} \partial Z^{k'}_{2,q}))$.

In the above conditions (i)–(vii), there appear κ^* 2-cells $Z^k_{1,p}$ $(1 \le k \le s_p,\ \mu_{\ell-1}+1 \le p \le \mu'_\ell,\ 0 \le \ell \le N)$ and $Z^k_{2,q}$ $(1 \le k \le t_q,\ \nu_{\ell-1}+1 \le q \le \nu'_\ell,\ 0 \le \ell \le N)$, where $\kappa^* = \sum_{\ell=0}^{N} (\sum_{p=\mu_{\ell-1}}^{\mu'_\ell} s_p + \sum_{q=\nu_{\ell-1}}^{\nu'_\ell} t_q)$. Besides them, there are $(m_0 - \kappa^*)$ 2-cells $Y'_{\kappa_*+1}, Y'_{\kappa_*+2}, \ldots, Y'_{m_0}$ ($Y'_{\kappa_*+i} = Y_k$ for some $1 \le k \le m_0$). Because of the condition (B.5), these 2-cells can be so ordered that, for each Y'_{κ^*+i}, we can find a 1-cell w'_i in $\mathcal{K}$ such that $w'_i \subset \partial Y'_{\kappa_*+i} \cap (\partial Y_0 \cup \tilde{Z}_{1,\mu_N} \cup \tilde{Z}_{2,\nu_N} \cup (\bigcup_{j=1}^{i-1} \partial Y'_{\kappa_*+j}))$. Moreover for each meridian disk $D_{1,p}$ $(\mu_N + 1 \le p \le g)$ we can take a 1-cell $v_{1,p}$ of $\mathcal{K}$ with $v_{1,p} \subset y_p$, and for each $D_{2,q}$ $(\nu_N + 1 \le q \le g)$ we can find a 1-cell $v_{2,q}$ with $v_{2,q} \subset z_q$.

Defining numbers $\lambda_1(\ell)$ and $\lambda_2(\ell)$ $(0 \le \ell \le N)$ by

$$\lambda_1(\ell) = \mu'_\ell + \nu'_\ell + \Big(\sum_{\ell'=1}^{\ell} \sum_{p=\mu_{\ell'-1}+1}^{\mu'_{\ell'}} s_p\Big) + \Big(\sum_{\ell'=1}^{\ell-1} \sum_{q=\nu_{\ell'-1}+1}^{\nu'_{\ell'}} t_q\Big),$$

$$\lambda_2(\ell) = \lambda_1(\ell) + (\mu_\ell - \mu'_\ell),$$

and numbers $\kappa_1(\ell,p)$ and $\kappa_2(\ell,q)$ $(1 \le \ell \le N,\ \mu_{\ell-1} \le p \le \mu'_\ell,\ \nu_{\ell-1} \le q \le \nu'_\ell)$ by

$$\kappa_1(\ell,p) = \lambda_2(\ell-1) + \Big(\sum_{p'=\mu_{\ell-1}+1}^{p} s_{p'}\Big) + (p - \mu_{\ell-1}),$$

$$\kappa_2(\ell,q) = \kappa_1(\ell,\mu'_\ell) + \Big(\sum_{q'=\nu_{\ell-1}+1}^{q} t_{q'}\Big) + (q - \nu_{\ell-1}),$$

we give an ordering to the meridian disks in the complete system $\vec{D}' = (D'_1, D'_2, \ldots, D'_{g'})$, which was introduced in Lemma 2.2, as follows.

- For $\lambda_1(\ell) + 1 \le i \le \lambda_1(\ell) + (\mu_\ell - \mu'_\ell)$ $(0 \le \ell \le N)$,
  $D'_i = D_{1,p} \cap H'_1$ $(i = \lambda_1(\ell) + (p - \mu'_\ell),\ \mu'_\ell + 1 \le p \le \mu_\ell)$.
- For $\lambda_2(\ell) + 1 \le i \le \lambda_2(\ell) + (\nu_\ell - \nu'_\ell)$ $(0 \le \ell \le N)$,
  $D'_i = D_{2,q} \cap H'_1$ $(i = \lambda_2(\ell) + (q - \nu'_\ell),\ \nu'_\ell + 1 \le q \le \nu_\ell)$.
- For $\kappa_1(\ell,p-1) + 1 \le i \le \kappa_1(\ell,p) - 1$ $(1 \le \ell \le N)$,
  $D'_i = Z^k_{1,p} \cap H'_1$ $(i = \kappa_1(\ell,p) + k,\ 1 \le k \le s_p,\ \mu_{\ell-1} + 1 \le p \le \mu'_\ell)$.
- For $i = \kappa_1(\ell,p)$ $(1 \le \ell \le N)$, $D'_i = D_{1,p} \cap H'_1$.
- For $\kappa_2(\ell,q-1) + 1 \le i \le \kappa_2(\ell,q) - 1$ $(1 \le \ell \le N)$,
  $D'_i = Z^k_{2,q} \cap H'_1$ $(i = \kappa_2(\ell,q) + k,\ 1 \le k \le t_q,\ \nu_{\ell-1} + 1 \le q \le \nu'_\ell)$.
- For $i = \kappa_2(\ell,q)$ $(1 \le \ell \le N)$, $D'_i = D_{2,q} \cap H'_1$.
- For $\kappa_2(N,\nu_N) + 1 \le i \le \kappa_2(N,\nu_N) + (m_0 - \kappa^*)$,
  $D'_i = Y'_{\kappa^*+j}$ $(i = \kappa_2(N,\nu_N) + j,\ 1 \le j \le m_0 - \kappa^*)$.
- For $\kappa_2(N,\nu_N) + (m_0 - \kappa^*) + 1 \le i \le \kappa_2(N,\nu_N) + (m_0 - \kappa^*) + (g - \mu_N)$,
  $D'_i = D_{1,p} \cap H'_1$ $(i = \kappa_2(N,\nu_N) + (m_0 - \kappa^*) + (p - \mu_N),\ \mu_N + 1 \le p \le g)$.
- For $\kappa_2(N,\nu_N) + (m_0 - \kappa^*) + (g - \mu_N) + 1 \le i \le g'$,
  $D'_i = D_{2,q} \cap H'_1$ $(i = \kappa_2(N,\nu_N) + (m_0 - \kappa^*) + (g - \mu_N) + (q - \nu_N),\ \nu_N + 1 \le q \le g)$.

For such an ordering we have that

Lemma 3.1. *There exists an ordered link $\mathcal{L}' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_{g'})$ in H'_1 such that*

(i) $\mathcal{L}'$ *is a* 0*-pseudo core which has* $\vec{D}'$ *as its associated meridian disk family,*

(ii) $\mathcal{L} \subset \mathcal{L}'$ *and* $\vec{\theta}$ *is a localizing arc system for* $\mathcal{L}'$, *where* $\mathcal{L}$ *and* $\vec{\theta}$ *are those in the condition* (B.6)$'$.

Proof. Using the same notation $\varepsilon(\cdot)$ as in the previous section, we define the components as follows.

- For $\lambda_1(\ell)+1 \le i \le \lambda_1(\ell)+(\mu_\ell-\mu'_\ell)$ $(0 \le \ell \le N)$,
  $\alpha'_i = \alpha_{1,p}$ $(i = \lambda_1(\ell)+(p-\mu'_\ell),\ \mu'_\ell+1 \le p \le \mu_\ell)$.
- For $\lambda_2(\ell)+1 \le i \le \lambda_2(\ell)+(\nu_\ell-\nu'_\ell)$ $(0 \le \ell \le N)$,
  $\alpha'_i = \alpha_{2,q}$ $(i = \lambda_2(\ell)+(q-\nu'_\ell),\ \nu'_\ell+1 \le q \le \nu_\ell)$.
- For $\kappa_1(\ell,p-1)+1 \le i \le \kappa_1(\ell,p)-1$ $(1 \le \ell \le N)$,
  $\alpha'_i = \varepsilon(u^k_{1,p})$ $(i = \kappa_1(\ell,p)+k,\ 1 \le k \le s_p,\ \mu_{\ell-1}+1 \le p \le \mu'_\ell)$.
- For $i = \kappa_1(\ell,p)$ $(1 \le \ell \le N)$, $\alpha'_i = \varepsilon(w_{1,p})$.
- For $\kappa_2(\ell,q-1)+1 \le i \le \kappa_2(\ell,q)-1$ $(1 \le \ell \le N)$,
  $\alpha'_i = \varepsilon(u^k_{2,q})$ $(i = \kappa_2(\ell,q)+k,\ 1 \le k \le t_q,\ \nu_{\ell-1}+1 \le q \le \nu'_\ell)$.
- For $i = \kappa_2(\ell,q)$ $(1 \le \ell \le N)$, $\alpha'_i = \varepsilon(w_{2,q})$.
- For $\kappa_2(N,\nu_N)+1 \le i \le \kappa_2(N,\nu_N)+(m_0-\kappa^*)$,
  $\alpha'_i = \varepsilon(w'_j)$ $(i = \kappa_2(N,\nu_N)+j,\ 1 \le j \le m_0-\kappa^*)$.
- For $\kappa_2(N,\nu_N)+(m_0-\kappa^*)+1 \le i \le \kappa_2(N,\nu_N)+(m_0-\kappa^*)+(g-\mu_N)$,
  $\alpha'_i = \varepsilon(v_{1,p})$ $(i = \kappa_2(N,\nu_N)+(m_0-\kappa^*)+(p-\mu_N),\ \mu_N+1 \le p \le g)$.
- For $\kappa_2(N,\nu_N)+(m_0-\kappa^*)+(g-\mu_N)+1 \le i \le g'$,
  $\alpha'_i = \varepsilon(v_{2,q})$ $(i = \kappa_2(N,\nu_N)+(m_0-\kappa^*)+(g-\mu_N)+(q-\nu_N),\ \nu_N+1 \le q \le g)$.

It is evident that $\mathcal{L}'$ is a link in H'_1 containing $\mathcal{L}$ as its sublink. It is also easy to see that the conditions for a reducing pair and the choice of 1-cells $u^k_{1,p}$, $u^k_{2,q}$, w_j *etc.* imply that $\mathcal{L}'$ is a 0-pseudo core with an associated meridian disk family $\vec{D}'$.

The link $\mathcal{L}[\vec{\theta}]$, which is obtained from $\mathcal{L}$ by cross changes along $\vec{\theta}$, is local by the condition (B.6)', and the components in $\mathcal{L}'-\mathcal{L}$, each of which is of the type $\varepsilon(\sigma)$ for some 1-cell σ, bound 2-disks which are mutually disjoint and do not intersect with $\vec{\theta}$. Hence the link $\mathcal{L}'[\vec{\theta}] = \mathcal{L}[\vec{\theta}] \cup (\mathcal{L}'-\mathcal{L})$ is also a local link. This completes the proof of Lemma 3.1. □

By the same way as the proof of Theorem 1.2, we can see that Lemma 3.1 and the condition (B.6)$'$ lead us to the conclusion of Theorem 1.4. □

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