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Combinatorial Classification of Genus 2 Three-dimensional Manifolds

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Introduction

It is one of the ultimate goals of topology to list up all the manifolds and distinguish them. This problem has been solved for some 100 years for one and two dimensional manifolds, namely, curves and surfaces. But the problem for three dimensional manifolds(3-manifolds) turns out to be extremely difficult and, in spite of very interesting and overview results, has not been solved yet. One of the reasons for this may be that it had long been not known a 'good' presentation for 3-manifolds which in turn gives a suitable mean for systematic classification.

DS-diagram is a tool to enumerate three-dimensional manifolds, which may be regarded as a generalization of the polygonal diagram which is very effective to classify two-dimensional manifolds, namely surfaces. Associated to DS-diagrams are E-data, which describe DS-diagrams as an arrangement of symbols and hence give a brief description of manifolds.

A complexity of E-data can be measured by the block number, which is naturally defined for every E-datum, and take non-negative integer values[1]. Hence corresponding DS-diagrams are also stratified according to their block numbers, and it is known that the minimum of the block numbers of a given manifold equals to its Heegaard genus. In particular it has been known that the manifolds of Heegaard genus 1, or equivalently which have block number-(less than-)1 DS-diagrams, are so-called Lens spaces and completely classified. Thus, a number of recent studies concentrate on classification of genus 2 three-dimensional manifolds.

In this thesis, we review these important notions (part I), and we focus on an unexplored class of E-data of block number 2, namely $\langle l,l,l,l;ad \mid cb\rangle$ in the notations of [4]; Earlier studies deals with $\langle l,l,l,l;ab \mid cd\rangle$ [7], $\langle r,l,r,l;cd|ab\rangle$ [4], for instance. For this class, we give a parametrization and under certain condition distinguished those which really represent manifolds (part II). Then for a specific series of manifolds we further investigate their geometric properties.

In particular, we explore their possible relations with simpler manifolds by branched coverings and also check their fiberedness. Main idea in investigating these problems is to deal with surfaces in a manifold, we rather deal with curves that occur at the intersection of surfaces and spines. By the so-called intersection numbers, most of these problems may be reduced to solving linear equations, at least partly. One of our results will show that a series of manifolds turns out to be the branched double cover of lens spaces. Finally, an invariant called the torsion linking form is applied to distinguish some manifolds.

Part I

Combinatorial presentation of manifolds

Chapter 1

DS-diagrams and E-data

Spines provide suitable basis for a generalization of the polygonal diagram for surfaces.

1.1 Spines

[1] Spines (in 3-manifolds), and in our context below, are singular surfaces which has been simplified as much as possible with respect to its singularity, without changing the topology of its complemet.

1.1.1 Polytope presentation

Definition 1 (CW-complex) *A set P is said to be a CW-complex if it is the union of sets $E^i (i = 0, 1, 2, \dots,)$,*

$$E^i := \{e^i_1, e^i_2, \dots\},$$

each of whose elements e^i_j is a map from the disk D^i onto P such that

$$e^i_j(\partial D^i) \subset E^{i-1}(P) \text{ and } e^i_j(t) \cap E^i(P) = \emptyset \text{ for } \forall t \in \mathrm{int}(D^i).$$

$$P := \bigcup_{i=0}^{\infty} E^i = \bigcup_i \bigcup_j e^i_j.$$

Let us call here a finite CW-complex, that is a finite set with respect to its cardinality as a polyhedron. And for 2-polyhedra Q, we also write the set E^0, E^1, E^2 as V, E, F respectively,

[1]We do not try to give the scattered original source for the contents of this section, instead generally refer to the concise article [1], where the original references may be found.

For the next definition, let Σ_x^+, Σ_y^- and Σ_z denote the following surfaces in $\mathbb{R}^3$;

$$\begin{aligned}
\Sigma_x^+ &:= \{(x,y,z) \in \mathbb{R}^3; x = 0, y \geq 0\} \\
\Sigma_y^- &:= \{(x,y,z) \in \mathbb{R}^3; y = 0, x \geq 0\} \\
\Sigma_z &:= \{(x,y,z) \in \mathbb{R}^3; z = 0\}
\end{aligned}$$

Definition 2 *Let M be a 3-manifold. A 2-polyhedron $P(\subset M)$ is said to be simple*[2] *if there is a neighborhood $N(x)$ (in M) for any point x of P which looks like one of the following three pictures:*

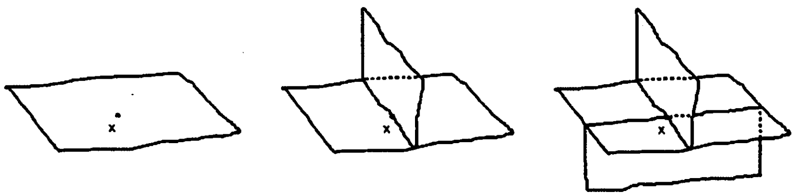


Figure 1.1: Local pictures of the neighborhood of a point $x \in P$.

In other words, there is a homeomorphism $h : N(x) \to \mathbb{R}^3$ such that

1. $h(N(x) \cap P) \subset \Sigma_z$.
2. $h(N(x) \cap P) \subset \Sigma_z \cup \Sigma_x^+$.
3. $h(N(x) \cap P) \subset \Sigma_z \cup \Sigma_x^+ \cup \Sigma_y^-$.

The set of all singular points of P, at which there is no $\mathbb{R}^2$-like neighborhood, is denoted by $S(P)$.

Proposition 1 ([5]) *Let M be a closed 3-manifold and $P(\subset M)$ be a simple polyhedron such that $M - P$ consists of some open 3-balls. Then the dual complex of P is a singular triangulation of M.*

Definition 3 *A 2-polyhedron P is called a special spine if $M - P$ is homeomorphic to an open 3-ball.*

Some of the examples in the previous chapter falls into this category.

Let P be a simple spine of M, and B its complementary 3-ball. The pair $(V(P), E(P))$ composes a 4-regular graph $S(P)$ in P. If we cut M along P, then $S(P)$ branches to be a 3-regular graph G on $\partial B \cong S^2$, as is seen by the local picture $fig.$2.1. Thus G naturally inherits graph structure (vertex, edge, faces) from $S(P)$; it preserves the 'vertex-edge-faces' stratification. Let f denote the inverse operation of this cutting, then $f(G) = S(P)$.

[2]Simple polyhedron is also called as a *fake surface*, see [?].

Definition 4 *The triad (B, G, f) is called a* <u>*DS-diagram*</u>*, obtained by the simple spine P, for the manifold M.*

Heegaard splitting for 3-manifolds ensures the existence of simple spines:

Theorem 1 *For an arbitrary 3-manifold, simple spine always exist.* ♠

Proof Let M be a closed 3-manifold, and $V_1 \cup_h V_2$ be a Heegaard-splitting for M, $\vec{D}_i = (D_{i1}, D_{i2}, \ldots, D_{ig})$ a complete meridian disk system for V_i $(i = 1, 2)$.[9]. Then we may always modify $\vec{D}_1$ by an isotopy of V_1 such that ∂D_{2j} crosses ∂D_{1k} transversally, or does not intersect at all, for $\forall j, k = 1, 2, 3, \ldots$.

Since V_i $(i = 1, 2)$ cut along $\vec{D}_i$ is a 3-ball B_i by definition, $M - (\partial V_1 \cup \vec{D}_1 \cup \vec{D}_2)$ consists of a pair of 3-balls. Choose a pair of faces σ_1, σ_2, and glue the two balls at the faces. On the boundary of the resulting one big 3-ball is there an attaching diagram. Therefore

$$P = \partial V_1 \cup \vec{D}_1 \cup_{\sigma_1 \sim \sigma_2} \vec{D}_2$$

gives a simple spine for M. □

Conversely, a three-tuple (B, G, f), which satisfies the same conditions as the DS-diagrams do, defines a closed 3-manifold.

Theorem 2 *Given a 3-ball B, a 3-regular graph G and a map $f : \partial B \to \partial B$ such that $f(\partial B)$ is a simple spine. Then the ball B whose boundary ∂B is identified by f defines a closed 3-manifold.* ♠

By virtue of this theorem, hereafter by a 'DS-diagram' we mean just the triad of the ball and the graph and the gluing map, and drop the manifold M in its original Definition 5.

Although DS-diagram should really be drawn spheres, in practice it is often written in the plane. This may be justified by removing a point which is disjoint from DS-diagrams, it may be identified with the plane.[3]

1.1.2 E-cycles, E-data and Arrangements

Definition 5 *A cycle γ of G is called an* <u>*E-cycle*</u> *of the DS-diagram (B, G, f) if*

1. *For every point $x \in P - S(P)$, $f^{-1}(x) \cap \gamma = \emptyset$.*
2. *For every point $x \in E(P)$, $f^{-1}(x) \cap \gamma$ consists of one and only one point, and the other two points of $f^{-1}(x)$ are in the opposite regions separated by γ.*
3. *For every point $x \in V(P)$, $f^{-1}(x) \cap \gamma$ consists of two and only two points, and the other two points of $f^{-1}(x)$ are in the opposite regions separated by γ.*

[3]The inverse of so-called stereographic projection.

In what follows, the ball B in (B, G, f) and an E-cycle γ are assumed to be given orientations in the following manner; We identify the ball B with the standard 3-ball $\{(x,y,z) \in \mathbb{R}^3; x^2+y^2+z^2 \leq 1\}$ such that the E-cycle γ coincides with the equator $\{z=0\} \cap \partial B$. Then the orientation of B induces an orientation for its boundary ∂B. Now we define the orientation of the E-cycle γ by the one induced by its upper hemisphere $\partial B \cap \{z > 0\}$.

We call the hemisphere $\partial B \cap \{z > 0\}$ or $B \cap \{z > 0\}$ as the upper hemisphere, north side or northern part, etc. Same for the opposite part. In this convention, four branched vertices which are identified as a single vertex v via f are distinguished as follows.

By definition of E-cycle, two of the preimage of v under f are contained in γ. The one which puts out an edge in the north (resp. south) side will be denoted by v^+ (resp. v^-). One of the preimages contained in the north (resp. south) side will be denoted by $\overline{v}$ (resp. $\underline{v}$).

Definition 6 *The cyclic order of the vertices $\{v^{\pm}; v \in V(G) \cap \gamma\}$ on the E-cycle γ is called an arrangement of $(B, G, f; \gamma)$.*

Theorem 3 ([1]) *Let (B, G, f) be a DS-diagram with an oriented E-cycle γ. The neighborhood of a vertex v on γ looks like either of the followings:*

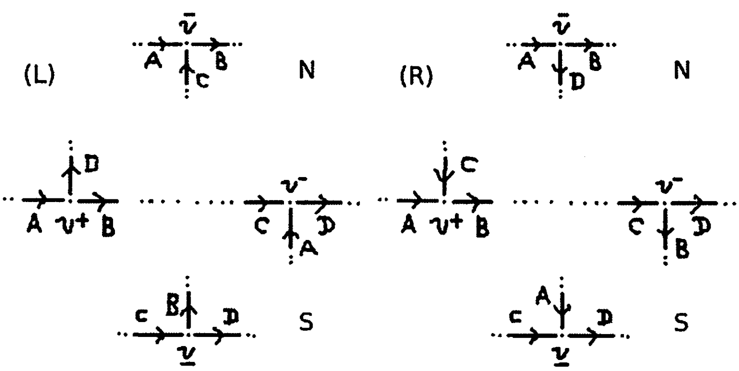


Figure 1.2: The horizontal line in the middle presents the E-cycle. The capital "N" and "S" indicate the northern part and southern part, respectively; (L) and (R) types defined in what follows below.

According to this theorem, every vertex is classified into either of the two types in the figure, and a vertex, the preimage of whose neighborhood looks like the picture in the left hand side of Figure 1.2 is said to be of (l)-type. If it looks like the picture in the right hand side, then is said to be of (r)-type.

Definition 7 *A map which assign a type (l) or (r) to each representative vertices $v \in V(G) \cap \gamma$ is called a type function.*

Let $\mathcal{A}$ denote the arrangement of $(B, G, f; \gamma)$, and ϕ a type function. The pair $(\mathcal{A}, \phi)$ is called an E-data for $(B, G, f; \gamma)$.

The values of type function, and hence E-data, are actually indicated by attaching the type values to each letter in the arrangement; for instance, v^+ of (l)-type in an arrangement is denoted by v^{+l}.[4]

Definition 8 *Let $\mathcal{E} = (\mathcal{A}, \phi)$ denote an E-data for a DS-diagram Δ. A block of consecutive positive (or negative) vertices in the arrangement $\mathcal{A}$ is called a positive (resp. negative) block. The number of positive blocks in $\mathcal{A}(\Delta)$ is called the block number of Δ, and is denoted as $bl(\Delta)$.*

Note that not all E-data may be realized as a complete DS-diagram.[1]
The minimum of the block numbers of all DS-diagrams which represent a given manifold is certainly an invariant of manifolds:

Definition 9 *Let M be a closed 3-manifold. The block number $Bl(M)$ of the manifold M is the minimum of the block numbers of all the diagrams which present M, i.e.*

$$Bl(M) = \min\{bl(\Delta) \mid M(\Delta) \cong M\}.$$

The following remarkable result has been established.

Theorem 4 ([4]) *For a closed 3-manifold M except S^3 and $S^2 \times S^1$, the block number equals to the Heegaard genus of M.* ♠

Remark 1 It is shown in [1] that $Bl(S^3) = 1$ and $Bl(S^2 \times S^1) = 0$.

1.1.3 Regular moves for E-data

Example 1 In the pictures below, a local picture of a special spine is shown. In it u, v are two consecutive vertices on the E-cycle. If we drag the vertical face along the oriented E-cycle so that the vertex u passes v, then it causes a certain transformation to the arrangement, evidently without changing the manifold it represents.
Noting that this deformation produces a new vertex x, the resulting E-datum is

$$u^{+l}v^{-l} \dots u^{-l} \dots v^{+l} \dots \Rightarrow v^{-l}u^{+l} \dots x^{-l}u^{-l} \dots v^{+l}x^{+l} \dots .$$

♣

As the above example shows, simple spines admit certain local deformations which does not affect the topology of its complement. The operations defined in the followings give, when combined, all sorts of such deformations.

Definition 10 *Let Δ be a DS-diagram with E-cycle and with an E-datum $\mathcal{E}$·. We define the following operations on Δ, which are called 'moves'.*

[4] However, if there is no fear of confusion, we often omit the signs and types, even overlines and underlines of vertices.

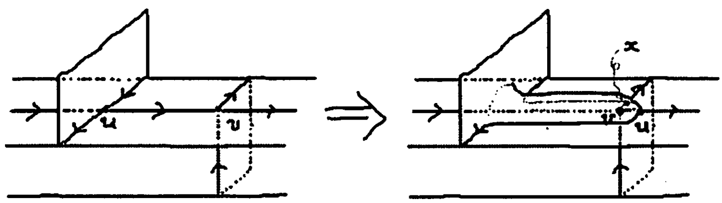


Figure 1.3: The blue line indicates the oriented E-cycle.

- *Suppose that $\mathcal{E}\Delta$ contains three subwords $W_1 = a^{-l}b^{+l}, W_2 = a^{+l}x^{+l}$ and $W_3 = x^{-l}b^{-l}$. Then the first regular move R_1 is the operation to replace these subwords $W_k (k = 1, 2, 3)$ by $W_1' = b^{+l}a^{-l}, W_2' = a^{+l}$ and $W_3' = b^{-l}$ respectively. The reversed operation, which is always possible, is denoted as R_1^{-1}.*
- *Suppose that $\mathcal{E}\Delta$ contains two subwords $W_1 = x^{-l}r^{-r}$ (or $y^{-r}x^{-l}$) and $W_2 = x^{+l}y^{+r}$. Then the second regular move R_2 is the operation simply to eliminate these subwords from the arrangement of $\mathcal{E}$. The reversed operation, if possible, is denoted as R_2^{-1}.*
- *Suppose that $\mathcal{E}\Delta$ contains a subword $W = x^{-l}y^{+l}z^{+l}z^{-l}w^{-r}y^{-l}w^{+r}x^{+l}$. Then surgery move S is the operation to eliminate this block from the arrangement of $\mathcal{E}$.*

The resulting diagram from each move $R_1^{\pm1}, R_2^{\pm1}, S^{\pm1}$, is denoted by $R_1^{\pm1}\Delta, R_2^{\pm1}\Delta, S^{\pm1}\Delta$ respectively. If two diagrams Δ and Δ' are related such that Δ' is obtained by a finite number of these moves on Δ, then they are said to be equivalent and is denoted as $\Delta \sim \Delta'$.

Theorem 5 *Let Δ and Δ' be DS-diagrams with E-cycle. Then there is an orientation-preserving homeomorphism from $M(\Delta)$ onto $M(\Delta')$ if and only if $\Delta \sim \Delta'$.* ♠

Definition 11 *A DS-diagram which can not be reduced in the block number by the second regular move is said to be irreducible.*

It will be useful to give the table of transformations which result from variants of R_1.

Proposition 2 *Let $\mathcal{E}$ be an E-data given as follows;*

$$\mathcal{E} = u^{+\phi(u)}v^{-\phi(v)}\ldots \tag{1.1.1}$$

If $\mathcal{E}$ is altered so that u^+ and v^- are swapped then u^- and v^+ will be transformed *in the following manner, depending on their types:*

$\phi(u)$	$\phi(v)$	u^-	v^+
l	l	$x^{-l}u^{-l}$	$v^{+l}x^{+l}$
l	r	$x^{-r}u^{-l}$	$x^{+r}v^{+r}$
r	l	$u^{-r}x^{-r}$	$v^{+l}x^{+r}$
r	r	$u^{-r}x^{-l}$	$x^{+l}v^{+r}$

In the case that the initial $\mathcal{E}$ begins with u^-v^+ and they be swapped, then the transformation rules are as follows:

$\phi(u)$	$\phi(v)$	u^+	v^-
l	l	$u^{+l}x^{+r}$	$x^{-r}v^{-l}$
l	r	$u^{+l}x^{+l}$	$v^{-r}x^{-l}$
r	l	$x^{+l}u^{+r}$	$x^{-l}v^{-l}$
r	r	$x^{+r}u^{+r}$	$v^{-r}x^{-r}$

Examples using this transformation will be found in the next chapter (cf. Example 7.)

1.2 Manifolds with block number 1

It is known that the only manifold which possesses the block number $Bl = 0$ is homeomorphic to $S^2 \times S^1$ [1]. Hence the first non-trivial problem is to characterize the manifold M with $bl(M) = 1$. Let Δ be an irreducible DS-diagram with an E-data, and have the block number 1.

In drawing DS-diagrams according to E-data, on account of theorem 5, it is convenient to draw the graph with some rigidity, in such a manner that consecutive edges whose corresponding edges on the E-cycle are also arranged continuously there, are drawn straight (or smooth), while the other (third) edges comes perpendicular to the straight line.

1.2.1 $\phi(a) = \phi(b) = l$

Let a^- and b^- denote two points on the E-cycle which bound the only positive block. Suppose there are p points in the block. Let α and β be the indexes of a^+ and b^+, respectively. Then one can draw the 'stalk' that grows from a^+ and that reaches to $\bar{b}$. This edge must be closed, since $\bar{b}$ appears somewhere middle in the edge and it has to inject to $\bar{b}$. Hence this edge encloses a region, but by the assumption that the diagram Δ is irreducible, it cannot contain any 2-gons inside the region. Therefore $\bar{b}$ should be adjacent to itself. And the rest of the

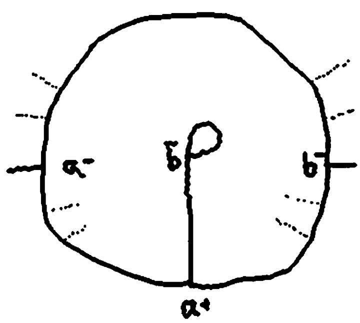


Figure 1.4: b is adjacent to itself.

diagram inside the E-cycle is uniquely determined.

To determine the order of negative points in the arrangement, let us trace the edge that starts from b^-.

Example 2 Let (B, G, f) be a DS-diagram with an E-datum

$$3^-1^+2^+3^+4^+4^- \cdots .$$

In the picture below, this E-datum is realized within the circle which E-cycle bounds. the edge on the E-cycle that starts from b^- is the same one which grows perpendicularly to E-cycle from b^+ (cf. Theorem 5), hence the vertex next to b^- is 1^-. In the same manner, the edge from the vertex 1^- is the same one that goes perpendicular to E-cycle from 1^+, thus the vertex next to 1^- is 2^-. Likewise the rest of the vertices in the negative block will be determined.

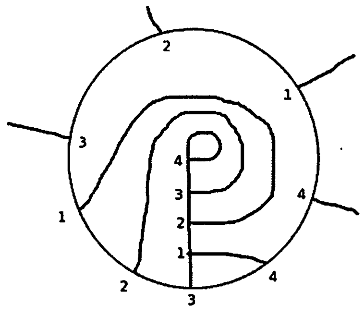
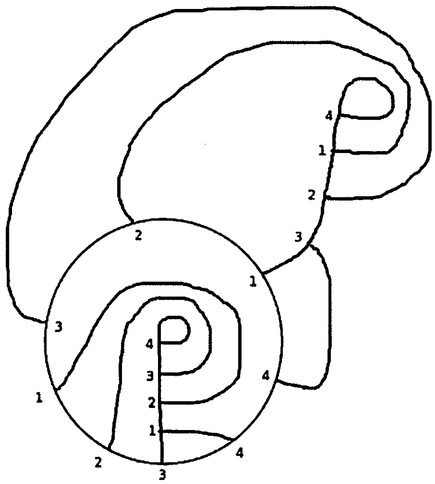


Figure 1.5: The order of the negative block is not determined a priori.

The complete DS-diagram is as follows



To determine the manifold it represents, it is convenient to construct the Heegaard diagram. Let C denote the curve which encloses the positive block as in the Figure 1.6 (The yellow-green curve). By splitting the ball B along this curve C, we get two balls. Each ball contains one pair of disks to be attached

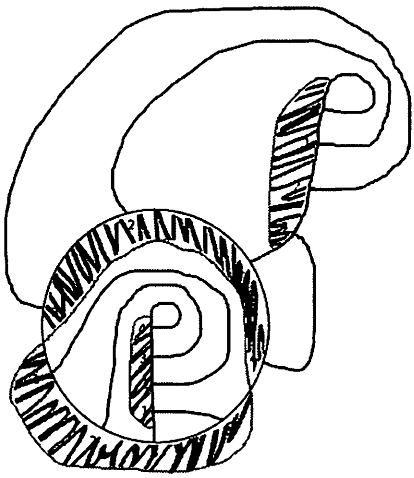


Figure 1.6: Splitting the ball along the disk which the yellow-green curve, we have a pair of two 3-balls.

together, thus represents a solid torus, hence gives a Heegaard splitting. Let V_1 denote the solid torus, obtained by attaching the northern hemisphere along the disks in pair, and V_2 denote the other solid torus. The meridian disk of V_2 is the region drawn with blue color in the Figure 1.6. Therefore the characteristic curve, is the periphery of this blue disk(Figure 1.7). We see that it is $(4,3)$-curve on ∂V_1, hence the manifold the diagram represents is the lens space $L(4,3)$.

♣

In a similar manner one can show that

Theorem 6 ([1]) *Let Δ be an irreducible diagram of block number 1, with an E-datum $\mathcal{E}$. If there are $2p$ vertices in $\mathcal{E}$, indexed as $1,2,3,\ldots,p$, and if $\mathcal{E}$ begins as*

$$q^{-l}1^{+l}2^{+l}\cdots p^{+l}p^{-l}\ldots,$$

then this E-datum is realizable and represents $L(p,q)$. ♠

Remark 2 It is shown in [1] that there is no irreducible diagram when $\phi(a) = l, \phi(b) = r$. The case $\phi(a) = r, \phi(b) = r$ is reduced to be the case $\phi(a) = \phi(b) = l$, because by reversing the orientation of the ball, the type of vertices are all inverted.

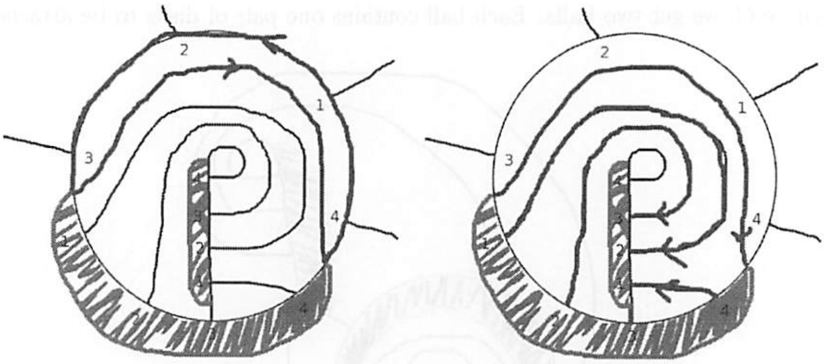


Figure 1.7: Left: The blue circle indicates the meridian disk of a solid torus. Right: The meridian curve drawn on the surface of the other solid torus

1.3 Fundamental Group

Fundamental groups may be readily calculated by looking at DS-diagrams.

Let P be a spine for the manifold M. Suppose there are n vertices in $S(P)$. Then the corresponding DS-diagram has $4n$ vertices and, being 3-regular graph, $6n$ edges. Therefore, on account of Euler's theorem, we find that there are $2n$ faces in the DS-diagram of M obtained by P.

Since $M-P$ is just a 3-ball B, gluing one pair of its faces (only their interiors, omitting boundaries) adds a generator each time, without any relation, to $\pi_1(B)$. Therefore $\pi_1(M-S(P))$ is a free group of rank $n+1$.

Relations occur when adding the remaining edges of $S(P)$, bringing contractible loops around them.

To each pair of faces assign numbers 1 to n. For a pair of faces indexed i, take an arc A_i which connects the faces inside the ball, oriented such that it goes from the face in the south to the other in the north. A_i will be a loop when glued, and let x_i denote its homotopy class.

As this local picture of spine shows, around each edge we arrive at

$$x_i x_j x_k^{-1} = 1$$

Basically, the $4n$ edges all yield some relations, however, there is much redundancy among them.

Example 3 ($L(5,4)$) Figure 2.10 displays the part of the DS-diagram of $L(p,q)$. Reading the relations of the regions at each edge on the E-cycle we obtain

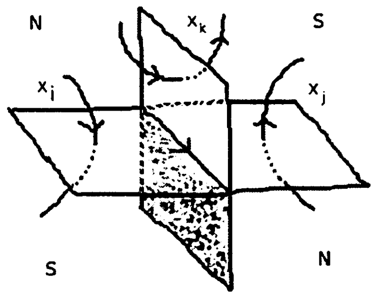


Figure 1.8: The edge with red arrow indicates the E-cycle and the dotted red plane an equatorial plane (a disk) which the E-cycle bounds in $M - P$. The capital "N" and "S" indicates the upper and the lower hemispheres.

$$
\begin{aligned}
\gamma_1\gamma_4 &= \gamma_5 \\
\gamma_2\gamma_4 &= \gamma_1 \\
\gamma_3\gamma_4 &= \gamma_2 \\
\gamma_4\gamma_4 &= \gamma_3 \\
\gamma_5\gamma_4 &= \gamma_4 \\
\gamma_1\gamma_4 &= \gamma_0 \\
\gamma_1\gamma_5 &= \gamma_1 \\
\gamma_1\gamma_1 &= \gamma_4 \\
\gamma_1\gamma_2 &= \gamma_4 \\
\gamma_1\gamma_3 &= \gamma_4.
\end{aligned}
\tag{1.3.1}
$$

It shows that

$$\gamma_1^5 = 1.$$

Thus, the diagram represented has the fundamental group $\mathbb{Z}/5\mathbb{Z}$. This is compatible with the theorem 6 which shows that the manifold represented is $L(5,4)(\simeq L(5,1))$.

♣

For large diagrams, it is convenient to take some maximal tree of $S(P)$. Since a tubular neighborhood T of the maximal tree is just a 3-ball, attaching T to $M - S(P)$ does not affect the fundamental group [10]. Therefore new relations only come from those edges which are not contained in T.

1.3.1 Fundamental group by dual complex

Spine is a dual of (singular) triangulation for a given manifold (Prop. 1). Hence a face of spine corresponds to an edge of its dual complex.

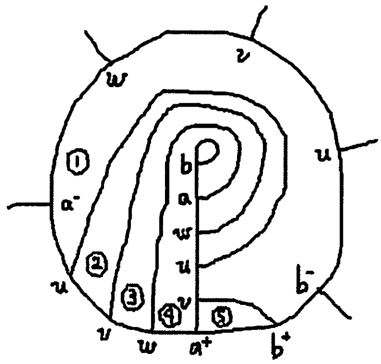


Figure 1.9: A part of the DS-diagram for $L(p, q)$ which interior to its E-cycle.

So one may use instead, the well-known presentation to obtain the group of a triangulated manifold. The procedure proceeds as follows:[10]

1. (Formally) take all the edges as generators, and consider the free group generated by these.
2. Take a maximal tree, and value all the edges belong to the tree as $= 1$.
3. Read relators from each face, i.e. the loop which encloses the face.

Example 4 (M_1 in §6.1.1) In the DS-diagram below, a maximal tree is indicated as the red edges. Omitting all edges of the maximal tree, the generators are

$$A, B, C, D, E, I, J, K, L.$$

And the relators are

0. E
1. AJ^{-1}
2. BK^{-1}
3. $CABCDEL^{-1}$
4. DA^{-1}
5. EIB^{-1}
6. JC^{-1}
7. $KIJKLD^{-1}$
8. LI^{-1}

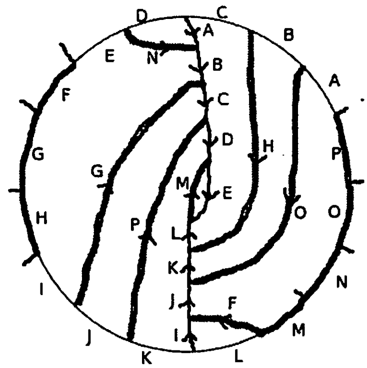


Figure 1.10: The union of all the edges colored red is a maximal tree.

Those 2- or 3-gon relations erase the redundant letters as

$$\begin{aligned} E &= 1 \\ C = D = J &= A \\ I = K = L &= B \end{aligned}$$

Thus remaining relators of number 3 and 7 now reduced to be

$$A^2BA^2B^{-1}$$
$$B^2AB^2A^{-1}.$$

This calculation will again be verified in the next chapter (Example 8).

♣

Example 5 ($L(p,q)$) Consider the irreducible DS-diagram (B,G,f) of $\mathrm{bl}=1$ for $L(p,q)$. Taking the whole consecutive edges in the negative block as a maximal tree, we have the relators from each face as

$$\begin{array}{rl} 1. & x_1x_qx_{q+1}^{-1} \\ 2. & x_2x_qx_{q+2}^{-1} \\ & \vdots \\ p-q. & x_{p-q}x_qx_p^{-1} \\ p-q+1. & x_{p-q+1}x_qx_1^{-1} \\ p-q+2. & x_{p-q+2}x_qx_2^{-1} \\ & \vdots \\ p. & x_px_qx_q^{-1} \\ p+1. & x_1x_qx_0^{-1} \end{array}$$

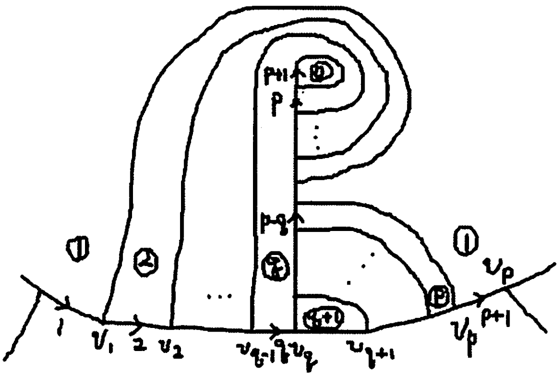


Figure 1.11: A picture of the positive block of the irreducible diagram of $L(p,q)$. A number in a circle indicates the index of the face.

We see that, $x_i x_q = x_r$, where $0 < r < p$ satisfies $r \equiv i + q \pmod p$. By the relation second from the bottom, $x_p = 1$, hence The order of x_i is the least positive integer k which satisfies

$$k \equiv -i\bar{q} \pmod p.$$

From this, we see that the only knot which bounds and which may be embedded as a simple arc in B are x_0 and x_p.

♣

Example 6 (Complement of a knot in $L(2,1)$) Let A_0 be a knot obtained by connecting the two 1-gons, one in the center, the other the farthest region outside.

$$A. \qquad x_1^2 = x_2 \qquad (1.3.2)$$

$$B. \qquad x_2 x_1 = x_1 \qquad (1.3.3)$$

$$C. \qquad x_1^2 = x_0 \qquad (1.3.4)$$

Excluding the third relation, we see that

$$\pi_1(S^3 - A_0) = \langle x_0, x_1 \mid x_1^2 \rangle \cong \mathbb{Z} * \mathbb{Z}_2.$$

♣

Part II

Genus-2 Manifolds

Chapter 2

Classification of the diagrams in the class $\langle l, l, l, l; ad|cb\rangle$

2.1 $\langle l, l, l, l; ad|cb\rangle$ with a bridge

Assuming only $\phi(a) = \phi(c) = l$ necessarily entails a 'bridged' arc, with which we mean the two 'stalks' which grow from a^{+l} and c^{+l} respectively meets and divide the diagram. We will assign parameters for DS-diagrams in this class as in the figure below; the restrictions for these parameters will be found to be

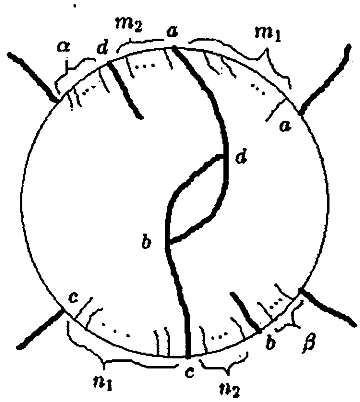


Figure 2.1: Parameters for the E-data are attributed as in this picture.

$$\begin{aligned} \alpha &= \beta \\ m_1 + 1 + m_2 &= m_2 + 1 + (\alpha - r) \\ n_1 + 1 + n_2 &= n_2 + 1 + (\beta - s) \end{aligned}$$

where r (or s) denote the number of arcs which bridges over two positive blocks in the left (or the right region, respectively); The first equation come from the 2-gon face in the center with vertices b, d. We find

$$\begin{aligned} \alpha &= r + s \\ m_1 - s &= n_1 - r \end{aligned}$$

Thus the simplest case when $\alpha = 0$, we have

$$r = s = 0, m_1 = n_1. \tag{2.1.1}$$

Here is the simplest diagram in this class.

Example 7 $(m_1 = m_2 = n_1 = n_2 = 0)$ The E-datum is

$$a^- a^+ d^+ b^- c^- c^+ b^+ d^-$$

This E-datum is transformed to be

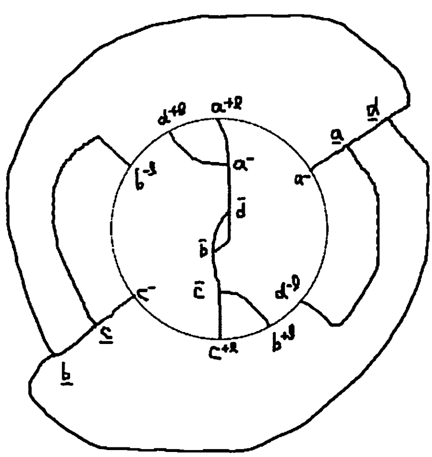


Figure 2.2: The DS-diagram of $M(0,0;0,0,0,0)$.

$$
\begin{aligned}
& a^+d^+b^-c^-c^+\underbrace{b^+d^-}a^- \\
\Rightarrow\ & a^+(d^+\epsilon_1^+)(\epsilon_1^-b^-)c^-\underbrace{c^+d^-}\underbrace{b^+a^-} \\
\Rightarrow\ & (a^+\epsilon_2^+)(d^+\delta_1^+)\epsilon_1^+\epsilon_1^-(\epsilon_2^-b^-)(\delta_1^-c^-)d^-\underbrace{c^+a^-}b^+ \\
\Rightarrow\ & (a^+\delta_2^+)\epsilon_2^+d^+\delta_1^+\epsilon_1^+\epsilon_1^-\epsilon_2^-b^-\delta_1^-(\delta_2^-c^-)d^-a^-c^+b^+ \\
\Rightarrow\ & (a^+\delta_2^+)\epsilon_2^+d^+\delta_1^+\epsilon_1^+\epsilon_1^-\epsilon_2^-b^-\delta_1^-(\delta_2^-c^-)d^-a^-c^+b^+
\end{aligned}
$$

Thus the resulting E-data

$$c^+b^+a^+\delta_2^+\epsilon_2^+d^+\delta_1^+\epsilon_1^+\epsilon_1^-\epsilon_2^-b^-\delta_1^-\delta_2^-c^-d^-a^-$$

is of block number 1, and from the theorem in §4, we see that the manifold it represents is $L(8,3)$.

$\clubsuit$

Let us adopt the following notation for the manifolds represented by the DS-diagrams we will consider:

Definition 12 *A manifold represented by the DS-diagram in the class $\langle l,l,l,l;ad \mid cb\rangle$ will be denoted by*

$$M(r,s;m_1,m_2,n_1,n_2).$$

2.2 $\alpha = 0$

In this case, we have, $r = s = 0$ and $m_1 = n_1$. Let p denote the value $m_1(= n_1)$.

2.2.1 $m_2 = 0, n_2 = 0$

One finds that he cannot draw a diagram for $p = 1$.

Example 8 ($p = 2$)

$$a^-x_1^+x_2^+a^+d^+b^-y_1^-x_2^-c^-y_1^+y_2^+c^+b^+d^-x_1^-y_2^-$$

To calculate the fundamental group, we take a maximal tree as the union of the edges with indices $6,7,9,13,14,15,16$, which is drawn red below in the figure 2.3. Then the relations are

$$
\begin{aligned}
x_1x_3 &= x_4 \\
x_2x_3 &= x_5 \\
x_3x_3 &= x_6 \\
x_4x_3 &= x_7 \\
x_5x_7 &= x_8 \\
x_6x_7 &= x_1 \\
x_7x_7 &= x_2 \\
x_8x_7 &= x_3
\end{aligned}
$$

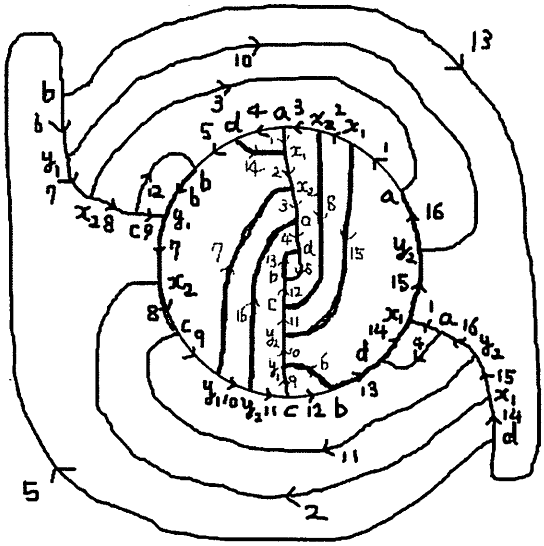


Figure 2.3: The DS-diagram for $M(0,0;2,0,2,0)$ with a maximal tree indicated in red color.

Eliminating all the x_i's except x_3 and x_7,

$$x_7 = x_3^2 x_7 x_3^2$$
$$x_3 = x_7^2 x_3 x_7^2.$$

Therefore the manifold represented, which we denote by M, has the fundamental group

$$\pi_1(M) = \langle x, y; x = y^2 x y^2, y = x^2 y x^2 \rangle.$$

The abelianized group is apparently $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, hence

$$H_1(M; \mathbb{Z}) = \mathbb{Z}_4 \oplus \mathbb{Z}_4.$$

♣

Example 9 ($p = 4$)

$$a^- x_1^+ x_2^+ x_3^+ x_4^+ a^+ d^+ b^- y_1^- x_2^- y_3^- x_4^- c^-$$
$$\text{-}y_1^+ y_2^+ y_3^+ y_4^+ c^+ b^+ d^- x_1^- y_2^- x_3^- y_4^-$$

A presentation of the fundamental group may be given by

$$\pi_1(M) = \langle x, y; x = y^2 xyxy^2, y = x^2 yxyx^2 \rangle.$$

$$H_1(M) = \mathbb{Z}_{24}.$$

♣

Remark 3 Notice that the graphs in the preceding two diagrams are rotation-symmetric, that is $\mathbb{Z}/2\mathbb{Z}$ acting on DS-diagram. There then expected to be a branched-covering of a sphere branched over two points (this is not obvious). Indeed in these cases they are, for there is an arc on the diagram which passes the edges $13, 16, 10, 16, 13$, in this order. This branched covering may be extended all over the ball it bounds, if its diameter between north and south poles is removed:

$$B^3 - \text{some diameter} \simeq (S^2 - \{0, \infty\}) \times (0, 1]$$

Therefore the manifolds are certain 2-fold branched coverings of the manifolds represented by the 'quotient' diagrams. These quotient diagrams are of block number 1, hence represent some lens spaces. Branched sets both upstairs and downstairs are knots. See §2, chapter 3 for details.

Proposition 3 *p can take only even integers.*

Proof Tracing the edges in the negative block from the vertex b^-, we find

$$b^- \to y_1^- \to x_2^- \to y_3^- \to x_4^- \to \cdots.$$

This alternating part of x and y terminates with x_{2n} or y_{2n} depending on p being even or odd. In the latter case, since the vertex a is adjacent to y_{2n}, a^- appears again in the negative block between the vertices b^- and c^-, but this is a contradiction. □

The former case, every such DS-diagram may be realized as shown in the figure 2.4 below.

Theorem 7 *Let $M_k := M(0,0;2k,0,2k,0)(k > 0)$[1] denote the manifold represented by the DS-diagram with an E-datum*

$$x_1^+ \ldots x_{2k}^+ d^+ \underbrace{b^- y_1^- x_2^- y_3^- x_4^- \cdots y_{2k-1}^- x_{2k}^- c^-}_{\textit{negative block}}$$

$$\text{-}y_1^+ y_2^+ \ldots y_{2k}^+ c^+ b^+ \underbrace{d^- x_1^- y_2^- x_3^- y_4^- \ldots x_{2k-1}^- y_{2k}^- a^-}_{\textit{negative block}}.$$

Then its fundamental group has the presentation

$$\pi_1(M_k) = \langle \alpha, \beta \mid \beta^2(\alpha\beta)^k\beta\alpha^{-1}, \alpha^2(\beta\alpha)^k\alpha\beta^{-1} \rangle.$$

Its homology group is,

$$H_1(M_k; \mathbb{Z}) = \begin{cases} \mathbb{Z}_{8k+8} & k \equiv 0, 2 \\ \mathbb{Z}_4 \oplus \mathbb{Z}_{2k+2} & k \equiv 1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{4k+4} & k \equiv 3 \end{cases},$$

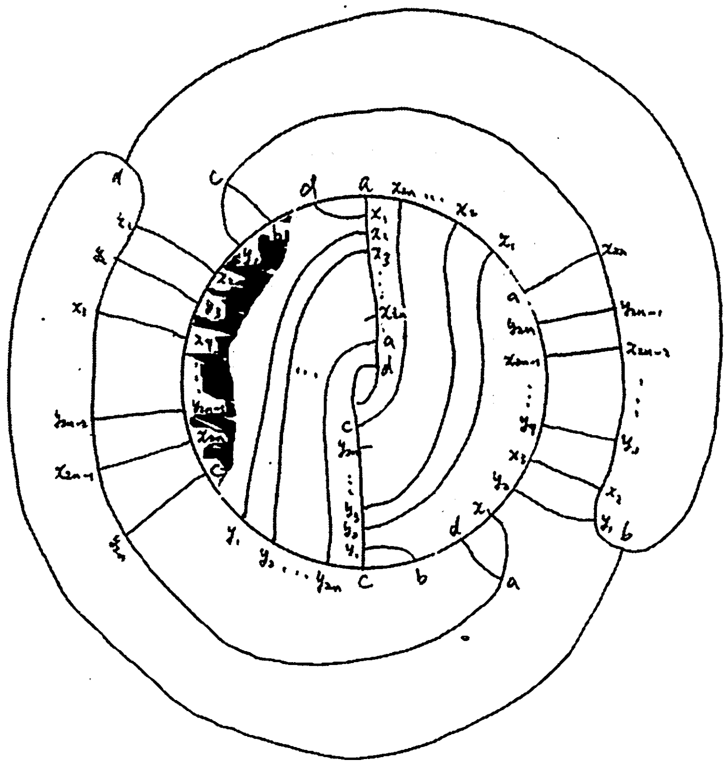


Figure 2.4: E-datum realized as a DS-diagram.

where the congruence should be understood in modulo 4. ♠

Proof To calculate the fundamental group, it is convenient to think of Heegaard diagram, as we have demonstrated in the example 2 for a lens space. We just have to read the homotopy type of the homotopy type of the circumference of the polygon which encloses the negative block (blue disk in Figure 2.4)

$$B \to b \xrightarrow{\beta} y_1 \xrightarrow{\beta} x_2 \xrightarrow{\alpha} y_3 \xrightarrow{\beta} \cdots \xrightarrow{\alpha} y_{2k-1} \xrightarrow{\beta} x_{2k} \xrightarrow{\alpha} c \xrightarrow{\beta} C \xrightarrow{\beta\alpha^{-1}} B\,,$$

where we understand, for example the path $y_i \to x_{i+1}$ is deformed to $y_i \to A \to x_{i+1}$. Hence we see $y_i \to x_{i+1} \to y_{i+2}$ contains a path homotopic to α. Therefore the whole loop is homotopic to

$$\beta^2(\alpha\beta)^k\beta\alpha^{-1}$$

[1]Remember that the case $k = 0$ is treated in Example 16. This notice applies for the following diagrams too, and will be omitted there.

and should be counted as a relator. Likewise the other relator is

$$\alpha^2(\beta\alpha)^k\alpha\beta^{-1}.$$

Abelianizing as $\alpha \mapsto a, \beta \mapsto b$, we have relations

$$(k-1)a + (k+3)b = 0 \tag{2.2.1}$$
$$(k+3)a + (k-1)b = 0. \tag{2.2.2}$$

Since the determinant of this presentation matrix is $8k+8$, any summands in a direct sum decomposition should be a subgroup of $\mathbb{Z}_{8k+8}$.

1. If k is even, then both (7.7) and (7.8) are irreducible relations. Subtracting, the upper from the lower, we have

$$4(a-b) = 0. \tag{2.2.3}$$

   Taking a new basis $(a', b') = (a-b, b)$ we have

$$4a' = 0 \tag{2.2.4}$$
$$(k-1)a' + (2k+2)b' = 0 \tag{2.2.5}$$

   Hence, for $k \equiv 0 \pmod 2$

$$H_1(M_k; \mathbb{Z}) = \mathbb{Z}_{8k+8}$$

2. If k is odd, and is the equations are reducible. Since the difference of the coefficients is 4, possible common divisor is 2 or 4.

   For the former, k should be congruent to 3 modulo 4; $k \equiv 3 \pmod 4$. Setting $k = 4l-1$ Then take a basis as $(a', b') = ((2l+1)a + (2l-1)b, (l+1)a + lb)$,

$$. \ 2a' = 0 \tag{2.2.6}$$
$$16lb' = (4k+4)b' = 0 \tag{2.2.7}$$

   Hence

$$H_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_{4k+4}.$$

   For the latter case, $k \equiv 1 \pmod 4$. Putting $a' = (a-b)$, one of the relations becomes, $4a' = 0$. Thus

$$H_1 = \mathbb{Z}_4 \oplus \mathbb{Z}_{2k+2}.$$

Collecting these the theorem is now established. □

2.2.2 $m_2 = 0, n_2 > 0$

Proposition 4 *$p \equiv (n_2+2) - 1$ is impossible.*

Proof Suppose p were congruent to -1 modulo $n_2 + 2$. Tracing the vertices in the negative block, we would have the sequence

$$b^- \to y^-_{n_2+1} \to x^-_{n_2+2} \to y^-_{2n_2+3} \to x^-_{2n_2+4} \to \cdots \to y_{p-1} \to a^-.$$

but this is a contradiction. □

Let us first study the cases when n_2 is even.

Example 10 ($n_2 = 2$) Let $M(p,2) = M(0,0;p,0,p,2)$ denote the manifold, and

$$\pi_1(M(p,2)) = \langle \alpha, \beta \mid r_1, r_2 \rangle$$

be its presentation.

1. If $p \equiv 0 \pmod 4$, put $p = 4k (k > 0)$

$$x_1^+ x_2^+ \dots x_{4k}^+ d^+$$
$$\text{-}b^- y_3^- x_4^- \cdots y_{4k-1}^- x_{4k}^- v_2^- y_2^- x_3^- \cdots y_{4k-2}^- x_{4k-1}^- v_1^- y_1^- x_2^- \cdots y_{4k-3}^- x_{4k-2}^- c^-$$
$$\text{-}\underbrace{y_1^+ y_2^+ \dots y_{3k}^+ c^+ v_1^+ v_2^+ b^+}_{\text{positive block}} d^- x_1^- y_4^- \dots x_{4k-3}^- y_{4k}^- a^-.$$

   The relators in the presentation of the fundamental group are

$$r_1 = (\beta(\beta\alpha)^k)^3 \beta^2 \alpha^{-1}$$
$$r_2 = \alpha(\alpha\beta)^k \alpha^2 \beta^{-1}.$$

   Its homology group is,

$$H_1(M(4k,2);\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_{9k+7}.$$

2. If $p \equiv 2 \pmod 4$, $k > 0$

$$x_1^+ x_2^+ \dots x_{4k+2}^+ d^+ \underbrace{b^- y_3^- x_4^- \cdots y_{4k-1}^- x_{4k}^- c^-}_{\text{negative block}} y_1^+ y_2^+ \cdots y_{4k+2}^+ c^+ v_1^+ v_2^+ b^+$$
$$\text{-}d^- x_1^- y_4^- \cdots x_{4k-3}^- y_{4k}^- x_{4k+1}^- v_1^- y_1^- x_2^- \cdots y_{4k+1}^- x_{4k+2}^-$$
$$\text{-}v_2^- y_2^- x_3^- \cdots y_{4k-2}^- x_{4k-1}^- y_{4k+2}^- a^-.$$

   The relators are

$$\begin{aligned} r_1 &= \beta(\beta\alpha)^k \cdot \beta^2 \alpha^{-1} \\ r_2 &= \alpha(\alpha\beta)^k \alpha\beta(\beta\alpha)^{k+1} \beta(\beta\alpha)^k \cdot \beta\alpha^2\beta^{-1}. \end{aligned}$$

   The first homology group is, for $k > 0$

$$H_1(M(4k+2,2);\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_{7k+9}.$$

   Note that, $M(2,2)$ is a Lens space $L(18,7)$.

♣

Remark 4 In the above example, we see that $M(4;2,0)$ and $M(6;2,2)$ have the same homology group. In fact there are infinitely many such pairs; they are $M(0,0;4(7i+1),0,4(7i+1),2)$ and $M(0,0;4(9i+1)+2,0,4(9i+1)+2,2)$, $i=0,1,2,\dots$. $i=0,1,2,\dots$. We will distinguish some of them in the next chapter.

Suppose n_2 is even. Set $2\nu = n_2+2$ and $k>0$. For $0<j\leq n_2 = 2\nu-2$, let m_j denote the maximal number such that

$$2m_j\nu + j < p.$$

Then, define a subword w_j for such j and m_j by

$$w_j := y_j^- x_{j+1}^- y_{2\nu+j}^- x_{2\nu+j+1}^- \cdots y_{2m_j\nu+j}^- x_{2m_j\nu+j+1}^-.$$

Example 11 ($n_2=4$) Let $M(-;p,0,p,4)$ denote the manifold.

1. If $p\equiv 0 \pmod 6$, put $p=6k (k>0)$.

$$x_1^+x_2^+\dots x_{6k}^+a^+d^+\underbrace{b^-w_5v_4^-w_4v_3^-w_3v_2^-w_2v_1^-w_1c^-}_{\text{negative block}}$$
$$-y_1^+y_2^+\dots y_{6k}^+c^+v_1^+v_2^+v_3^+v_4^+b^+\underbrace{d^-x_1^-w_6y_{6k}^-a^-}_{\text{negative block}}$$

$$\begin{aligned} r_1 &= (\beta(\beta\alpha)^k)^5\beta^2\alpha^{-1} \\ r_2 &= \alpha^2(\beta\alpha)^{k-1}\beta\alpha\alpha\beta^{-1} \end{aligned}$$

Its homology group is,

$$H_1(M;\mathbb{Z}) = \begin{cases} \mathbb{Z}_{28k+20} & k\equiv 0,1,3 \pmod 4 \\ \mathbb{Z}_4\oplus\mathbb{Z}_{7k+5} & k\equiv 2 \end{cases}.$$

2. If $p\equiv 4 \pmod 6$, put $p=6k+4 (k\geq 0)$.

$$x_1^+x_2^+\dots x_{6k+4}^+a^+d^+\quad\underbrace{b^-w_5c^-}_{\text{negative block}}$$
$$-y_1^+y_2^+\dots y_{6k+4}^+c^+v_1^+v_2^+v_3^+v_4^+b^+\underbrace{d^-x_1^-w_6v_1^-w_1v_2^-w_2v_3^-w_3v_4^-w_4y_{6k+4}^-a^-}_{\text{negative block}}$$

$$\begin{aligned} r_1 &= \beta(\beta\alpha)^k\beta^2\alpha^{-1} \\ r_2 &= \alpha^2(\beta\alpha)^k(\beta(\beta\alpha)^{k+1})^4\alpha\beta^{-1} \end{aligned}$$

The first homology group is

$$H_1(M;\mathbb{Z}) = \begin{cases} \mathbb{Z}_{20k+28} & k\equiv 0,1,3 \pmod 4 \\ \mathbb{Z}_4\oplus\mathbb{Z}_{5k+7} & k\equiv 2 \end{cases}.$$

♣

The following fact will be observed;

Proposition 5 *If n_2 is even, possible p are only those which are not congruent, modulo n_2+2, to odd numbers nor to any divisor of n_2+2.*

Proof Let $2\nu = n_2+2$. Define sequence of integers $\{\lambda_i; i \in \mathbb{N}\cup\{0\}, 0 < \lambda_i \leq 2\nu-2\}$ by

$$\lambda_0 \equiv -1 \tag{2.2.8}$$
$$\lambda_{i+1} \equiv \lambda_i - (\rho+1) \pmod{2\nu}. \tag{2.2.9}$$

Then we find

$$\lambda_i = -1 - i(\rho+1) \tag{2.2.10}$$

for $\forall i > 0$. Define a subword w_i for $0 < i \leq 2\nu$ by

$$w_i = y_i x_{i+1} y_{2\nu+i} x_{2\nu+i+1} \cdots y_{2\nu m_i+i} x_{2\nu m_i+i+1}$$

where m_i denotes the maximal integer which satisfies $2\nu m_i + i + 1 \leq p$. Then we may express the negative block in the E-data as follows

$$b^- w_{2\nu-1} v_{\lambda_1} w_{\lambda_1} v_{\lambda_2} w_{\lambda_3} \cdots \tag{2.2.11}$$

The negative block closes if it comes to the vertex c^-, whence the λ should be $\equiv \rho - 1$: Suppose m is the least integer such that $\lambda_m \equiv -1$. Then we have

$$(m+1)(\rho+1) \equiv 0 \pmod{2\nu} \tag{2.2.12}$$

On the other hand, it is a contradiction if the negative block contains a^-, which indeed happens in the following case: Let n be an integer such that $\lambda_n \equiv \rho - 1$. Then

$$(n+1)(\rho+1) \equiv 1 \pmod{2\nu} \tag{2.2.13}$$

Therefore, if $m > n$ for certain ρ, then the corresponding E-data cannot be realized. We easily see that, if $\rho+1$ divides 2ν there is no such n, hence $m < n$ in this case. if $(\rho+1, 2\nu) = d$, then

$$(m+1)\frac{\rho+1}{d} \equiv 0 \pmod{\frac{2\nu}{d}} \tag{2.2.14}$$
$$(n+1)\frac{\rho+1}{d} \equiv 1 \pmod{\frac{2\nu}{d}}. \tag{2.2.15}$$

The solutions are $m \equiv -1$ and $n \equiv ((\rho+1)/d)^{-1}$. In any case, m is the largest as an representative element of residue classes $\{a; (a, 2\nu/d) = 1\}$. Hence we have $m > n$, thus leading the corresponding E-data inconsistent. The realizability of the E-data corresponding to the 'sifted' p will be shown in the proof of the next theorem. $\square$

We now give the general result. Note that we write $v_{2\nu-1}$ instead of b.

Theorem 8 *Let $2\nu = n_2 + 2$. Assume p satisfies the conditions in the proposition 5. The general form of realizable E-data $M(0,0;p,0,p,2\nu-2)$ is*

$$x_1^+ x_2^+ \dots x_p^+ a^+ d^+ \; W_1 \; y_1^+ y_2^+ \dots y_p^+ c^+ v_1^+ v_2^+ \dots v_{2\nu-1}^+ \; W_2$$

where $W_i (i = 1,2)$ denotes a negative block.

$$\begin{aligned} W_1 &= v_{2\nu-1} w_{\lambda_1} v_{\lambda_2} w_{\lambda_2} \dots v_{\lambda_m} w_{\lambda_m} c^- \\ W_2 &= d^- x_1^- w_{\lambda'_1} v_{\lambda'_2} w_{\lambda'_2} \dots v_{\lambda'_n} w_{\lambda'_n} a^- \end{aligned}$$

where the sequences $\{\lambda\}, \{\lambda'\}$ and integers m, n are those defined in the proposition 5. ♠

Proof We only show the realizability of the diagram. For a subword w_j, the corresponding edge outside the E-cycle is denoted by $\underline{w}_j$, for example

$$\underline{w}_j = \underline{y}_j \underline{x}_{j+1} \underline{y}_{j+2\nu} \underline{x}_{j+2\nu+1} \cdots .$$

Define a sequence $\{\lambda'_i\}$ as the one which satisfies the same recursive relation as the equation (2.2.9), with the initial value $\lambda'_0 = 2\nu$. Then, by the preceding proposition, the negative blocks may be expressed as

$$\begin{aligned} P_1 &= b^- W_{\lambda_0} W_{\lambda_1} \dots W_{\lambda_m} \\ P_2 &= d^- x_1^- W_{\lambda'_0} W_{\lambda'_1} \dots W_{\lambda'_n} \end{aligned}$$

where m (or n) denotes the least integer such that $\lambda_m \equiv -1 \pmod{2\nu}$ (or $\lambda'_n \equiv p \pmod{2\nu}$ respectively). Let α denotes the index such that $\lambda_\alpha \equiv 1$. From this

$$\lambda_{\alpha+i} = \lambda'_i + 1. \tag{2.2.16}$$

for $\lambda'_0 \equiv 2\nu \equiv \lambda_\alpha - 1$. In the backward direction, since $\lambda_m \equiv p+1$ and $\lambda_{\alpha-1} \equiv 1 + (p+1)$, we also have

$$\lambda_{\alpha-j} = \lambda_{m-j} + 1 \tag{2.2.17}$$

for $j \geq \alpha$. Furthermore, since $\lambda_{m-\alpha-1} = \lambda_{m-\alpha} + (p+1) = \lambda_0 + p = 2\nu - 1 + p \equiv \lambda'_n - 1$, we have for $i < m - \alpha$

$$\lambda_{m-\alpha-i} + 1 = \lambda'_{n-i}. \tag{2.2.18}$$

Let $\underline{P}_1, \underline{P}_2$ denote the block and edges outside the E-cycle which corresponds to P_1, P_2 respectively on the E-cycle. From the theorem 5, we see that they starts from the vertices y_1^-, x_1^-. We now see, that in light of equations (2.2.16-18), the blocks on $\underline{P}_i$ all close. A part of this situation is drawn in the picture below. □

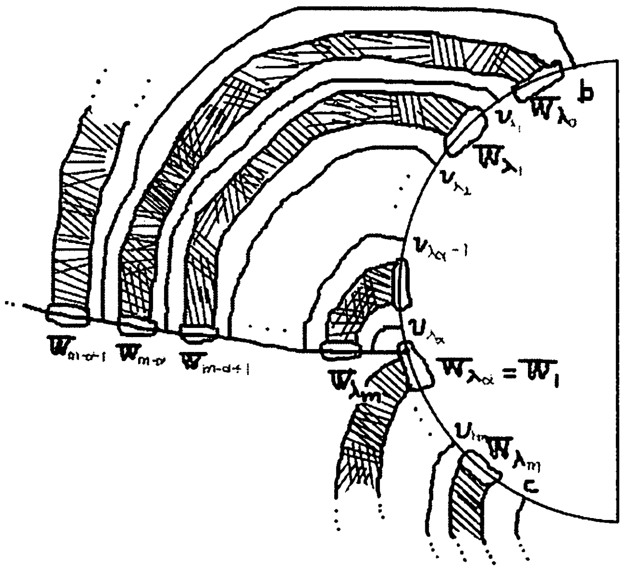


Figure 2.5: All the negative blocks W's close evenly with the corresponding blocks outside the E-cycle.

Remark 5 Observe that when $p \equiv 0 \pmod{2\nu}$, W_2 does not contain any v_i. Hence we can transform the diagram to a symmetric one

$$a^- x_1^+ x_2^+ \dots x_p^+ a^+ \epsilon^+ d^+ W_1' y_1^+ y_2^+ \dots y_p^+ c^+ v_1^+ v_2^+ \dots v_{\nu-1}^+ W_2'$$

which is a class of $M(0,0;p,\nu-1,p,\nu-1)$

$$M(0,0;p,0,p,2k\nu) \mapsto M(0,0;p+k+1,\nu-1,p+k+1,\nu-1)$$

$$x_1^+ x_2^+ \dots x_{6k}^+ a^+ d^+ \underbrace{b^- w_5 v_4^- w_4 v_3^- w_3 v_2^- w_2 v_1^- w_1 c^-}_{\text{negative block}}$$
$$\text{-}y_1^+ y_2^+ \dots y_{6k}^+ c^+ v_1^+ v_2^+ v_3^+ v_4^+ b^+ \underbrace{d^- x_1^- w_6 y_{6k}^- a^-}_{\text{negative block}}$$

And the diagram is necessarily a double cover of the lens space.

If we move the first positive block to the second one, we will have a diagram of $\langle l,l,l,l; \mid adcb\rangle$, or equivalently of $\langle l,l,l,l; cbad \mid\rangle$.

2.2.3 When n_2 is odd

Example 12 ($n_2 = 1$) There are two classes as in the table.

1. if $p \equiv 0 \pmod 3$, settting $p = 3k (k > 0)$, realizable E-data are

$$\overbrace{x_1^+ x_2^+ \cdots x_{3k}^+ d^+}^{\text{positive block}} \underbrace{b^- y_2^- x_3^- \cdots y_{3k-1}^- x_{3k}^- v^- y_1^- x_2^- \cdots y_{3k-2}^- x_{3k-1}^- c^-}_{\text{negative block}}$$

$$\text{-} \overbrace{y_1^+ y_2^+ \cdots y_{3k}^+ c^+ v^+ b^+}^{\text{positive block}} d^- x_1^- y_3^- \cdots x_{3k-2}^- y_{3k}^- a^-.$$

$$\begin{aligned} r_1 &= (\beta(\beta\alpha)^k)^2 \beta^2 \alpha^{-1} \\ r_2 &= \alpha^2 (\alpha\beta)^k \alpha\alpha\beta^{-1}. \end{aligned}$$

Its homology group is,

$$H_1(M_{1,1}; \mathbb{Z}) = \begin{cases} \mathbb{Z}_3 \oplus \mathbb{Z}_{5k+5} & k \equiv 0 \pmod 3 \\ \mathbb{Z}_{15k+15} & k \equiv 1, 2 \pmod 3 \end{cases}.$$

2. if $p \equiv 1 \pmod 3$, setting $p = 3k + 1 (k \geq 0)$, realizable E-data are

$$a^- \overbrace{x_1^+ x_2^+ \cdots x_{3k+1}^+ d^+}^{\text{positive block}} \underbrace{b^- y_2^- x_3^- \cdots y_{3k-1}^- x_{3k}^- c^-}_{\text{negative block}}$$

$$\text{-} \overbrace{y_1^+ y_2^+ \cdots y_{3k+1}^+ c^+ v^+ b^+}^{\text{positive block}} d^- x_1^- y_3^- \cdots x_{3k-2}^- y_{3k}^- x_{3k+1}^- v^- y_1^- x_3^- \cdots y_{3k-2}^- x_{3k}^- y_{3k+1}^-.$$

Then its fundamental group has the presentation

$$\pi_1(M_k) = \langle \alpha, \beta; r_1, r_2 \rangle$$

where the relators are

$$\begin{aligned} r_1 &= \beta^2 (\alpha\beta)^k \beta \alpha^{-1} \\ r_2 &= \alpha^2 \{\beta(\alpha\beta)^k\}^2 \alpha^2 \beta^{-1} \end{aligned}$$

Its homology group is,

$$H_1(M_1; \mathbb{Z}) = \begin{cases} \mathbb{Z}_4 \oplus \mathbb{Z}_6 & k = 1 \\ \mathbb{Z}_{11k+13} & k = 0, 2, 3, 4, \cdots \end{cases}.$$

♣

One can verify these in much the same manner as in the theorem 5 above.

Remark 6 Note that $\pi_1(M_0(1,1)) = \mathbb{Z}_{13}$, and hence the manifold is some Lens space. Successive application of the first regular move shows that it is in fact $L(13,8) \simeq L(13,5)$.[2]

Example 13 ($n_2 = 3$) Let $\rho(0 < \rho < 4)$ denote the residue of p modulo 5. The following E-data are realizable and represent certain manifolds.

$$V_1^+ W_1^- V_2^+ W_2^-,$$

where the positive blocks V_1^+ and V_2^+ are ($\pm$ omitted for brevity, same in the followings)

$$\begin{aligned} V_1^+ &= x_1 x_2 \cdots x_p a d \\ V_2^+ &= y_1 y_2 \cdots y_p c u_1 u_2 u_3 b. \end{aligned}$$

and the negatives W_1 and W_2 are

p (mod 5)	W_1^-	W_2^-
0	$by_4x_5\cdots y_{5k-1}x_{5k}u_3y_3x_4\cdots y_{5k-2}x_{5k-1}$ $u_2y_2x_3\cdots y_{5k-3}x_{5k-2}u_1y_1x_2\cdots y_{5k-4}x_{5k-3}c$	$dx_1y_5\cdots x_{5k-4}y_{5k}a$
1	$by_4x_5\cdots y_{5k-1}x_{5k}u_2y_2x_3\cdots y_{5k-3}x_{5k-2}c$	$dx_1y_5\cdots x_{5k-4}y_{5k}x_{5k+1}u_3y_3x_4\cdots y_{5k-2}x_{5k-1}$ $u_1y_1x_2\cdots y_{5k-4}x_{5k-3}y_{5k+1}a$
2	$by_4x_5\cdots y_{5k-1}x_{5k}u_1y_1x_2\cdots y_{5k+1}x_{5k+2}$ $u_3y_3x_4\cdots y_{5k-2}x_{5k-1}c$	$dx_1y_5\cdots x_{5k-4}y_{5k}x_{5k+1}$ $u_2y_2x_3\cdots y_{5k-3}x_{5k-2}y_{5k+2}a$
3	$by_4x_5\cdots y_{5k-1}x_{5k}c$	$dx_1y_5\cdots x_{5k-4}y_{5k}x_{5k+1}u_1y_1x_2\cdots y_{5k+1}x_{5k+2}$ $u_2y_2x_3\cdots y_{5k+2}x_{5k+3}u_3y_3x_4\cdots y_{5k-2}x_{5k-1}y_{5k+3}a$

Let us denote the manifold represented by an E-datum with $n_2 = 3, \rho \equiv p$ (mod 5) in the above table by $M_{3,\rho}$. Then a finite representation for its fundamental group may be given by

$$\pi_1(M_{3,\rho}) = \langle \alpha, \beta; r_1, r_2 \rangle$$

p (mod 5)	r_1	r_2
0	$(\beta(\beta\alpha)^k)^4\beta^2\alpha^{-1}$	$\alpha(\alpha\beta)^k\alpha^2\beta^{-1}$
1	$(\beta(\beta\alpha)^k)^2\beta^2\alpha^{-1}$	$\alpha(\alpha\beta)^{k+1}(\beta(\beta\alpha)^k)^2 \cdot \beta\alpha^2\beta^{-1}$
2	$(\beta(\beta\alpha)^k)\beta(\beta\alpha)^{k+1}(\beta(\beta\alpha)^k)\beta^2\alpha^{-1}$	$\alpha(\alpha\beta)^k\alpha\beta(\beta\alpha)^k \cdot \beta\alpha^2\beta^{-1}$
3	$(\beta(\beta\alpha)^k)\beta^2\alpha^{-1}$	$\alpha(\alpha\beta)^k\alpha(\beta(\beta\alpha)^{k+1})^2\beta(\beta\alpha)^k \cdot \beta\alpha^2\beta^{-1}$

Its homology group is,

p (mod 5)	$H_1(M_{3,\rho};\mathbb{Z})$
0	$\mathbb{Z}_{23k+17}$
1	$\mathbb{Z}_{51k+57}$
2	$\mathbb{Z}_3 \oplus \mathbb{Z}_{21k+24}$
3	$\mathbb{Z}_{17k+23}$

[2] If we try to move b^{+l} to the positive block along the orientation of E-data, m_2 will increase by 1.

♣

Theorem 9 *Let $2\nu - 1 = n_2 + 2$. The general form of realizable E-dataum $M(0,0;p,0,p,2\nu-3)$ is, for $p \not\equiv -1 \pmod{2\nu - 1}$*

$$x_1^+ x_2^+ \dots x_p^+ a^+ d^+ \; P_1 \; y_1^+ y_2^+ \dots y_p^+ c^+ v_1^+ v_2^+ \dots v_{2\nu-3}^+ b^+ \; P_2,$$

where $P_i (i = 1,2)$ denotes a negative block. To describe them explicitly, we further define λ_i and $\lambda'_i (0 \le \lambda_i, \lambda'_i \le 2\nu - 1)$ for $i > 0$ by the same recursive formula

$$\lambda_{i+1} \equiv \lambda_i - p - 1 \pmod{2\nu - 1}.$$

with the initial values $\lambda_0 = 2\nu - 1, \lambda'_0 = 2\nu$. Then

$$\begin{aligned} P_1 &= b^- W_{\lambda_0}^- v_{\lambda_1}^- W_{\lambda_1} \dots v_\lambda^- W_\lambda c^- \\ P_2 &= d^- x_1^- W_{\lambda'_0} v_{\lambda'_1} W_{\lambda'_1} \dots v_{\lambda'} W_{\lambda'} y_p^- a^- \end{aligned}$$

where λ (or λ', respectively, same in the following) denotes the first member in $\{\lambda_i\}$ (or $\{\lambda'_i\}$) such that $\lambda \equiv -(2\nu - 3) - 1$ (or $\lambda' \equiv p$). ♠

Proof The proof will be a special case of that of theorem 10 in the next section. Hence omitted here. □

Remark 7 (Hierarchy problem) In the above diagram, if we move b^+, by the regular move (I), to the positive block which contains a^+, at first sight the diagram belong to the former case we have studied. However it turns out this operation increases m_2, which is not captured within our scope so far, as shown below. if $n_2 = 1$ and $p = 3$,

$$\begin{aligned} & \dots y_3^{+l} c^{+l} u^{+l} \underbrace{b^{+l} d^{-l}} x_1^- y_3^- a^- x_1^+ x_2^+ x_3^+ a^+ d^+ b^- y_2^- \dots \\ \Leftrightarrow \quad & \dots y_3^{+l} c^{+l} u^{+l} d^{-l} \underbrace{b^{+l} x_1^-} y_3^- a^- x_1^{+l} x_2^+ x_3^+ a^+ \underline{d^+ \epsilon_1^{+l} \epsilon_1^{-l} b^{-l}} y_2^{-l} \dots \\ \Leftrightarrow \quad & \dots y_3^{+l} c^{+l} u^{+l} d^{-l} x_1^{-l} \underbrace{b^{+l} y_3^-} a^- \underline{x_1^{+l} \epsilon_2^{+l}} x_2^{+l} x_3^{+l} a^{+l} d^+ \epsilon_1^{+l} \epsilon_1^{-l} \underline{\epsilon_2^{-l} b^{-l}} y_2^{-l} \dots \\ \Leftrightarrow \quad & \dots \underline{y_3^{+l} \epsilon_3^{+l}} c^{+l} u^{+l} d^{-l} x_1^{-l} y_3^{-l} \underbrace{b^{+l} a^-} x_1^{+l} \epsilon_2^{+l} x_2^{+l} x_3^{+l} a^{+l} d^+ \epsilon_1^{+l} \epsilon_1^{-l} \epsilon_2^{-l} \underline{\epsilon_3^{-l} b^{-l}} y_2^{-l} \dots \\ \Leftrightarrow \quad & \dots y_3^{+l} \epsilon_3^{+l} c^{+l} u^{+l} \underbrace{d^{-l} x_1^{-l} y_3^{-l} a^-}_{\text{neg.}} \underbrace{b^{+l} x_1^{+l} \epsilon_2^{+l} x_2^{+l} x_3^{+l} \underline{a^{+l} \epsilon_4^{+l}} d^+ \epsilon_1^{+l}}_{\text{pos.}} \epsilon_1^{-l} \epsilon_2^{-l} \epsilon_3^{-l} \underline{\epsilon_4^{-l} b^{-l}} y_2^{-l} \dots \end{aligned}$$

Thus the resulting code is

$$y_1^+ y_2^+ y_3^+ \epsilon_3^+ c^+ u^+ d^- x_1^- y_3^- a^- b^+ x_1^+ \epsilon_2^+ x_2^+ x_3^+ a^+ \epsilon_4^+ d^+ \epsilon_1^+$$

and this is a diagram with $p = 4$ and $m_2 = 1, n_2 = 0$.

Suppose the last half of the positive block between c^- and d^- is read as $\dots c^+ u_1^+ u_2^+ \dots u_\mu^+ b^+$. It suffices to prove that we may always move u_i^+'s and b^+. However, when $m_2 + n_2$ is an even number we find it convenient to have m_2 and n_2 equal, in order to symmetrize the diagram.

2.3 $m_2 > 0$

By obvious symmetry, if $n_2 = 0$, it is actually the same E-data as of $m_2 = 0$. So let us start with $n_2 = 1$.

Example 14 ($n_2 = 1$) $M(0,0;2k,1,2k,1)$ is realizable.

$$x_1^+x_2^+\dots x_{2k}^+a^+u^+d^+\text{-}b^-y_2^-x_4^-\dots y_{2k}^-u^-x_1y_3\dots x_{2k-1}c^-$$
$$\text{-}y_1^+y_2^+\dots y_{2k}^+c^+v^+b^+d^-x_2^-y_4^-\dots x_{2k}v^-y_1^-x_3^-\dots y_{2k-1}^-a^-$$

♣

Example 15 ($n_2 = 2$) $M(0,0;k,1,k,2)$ is realizable.

$$x_1^+x_2^+\dots x_{3k}^+a^+u^+d^+b^-y_2x_4\dots y_dw_3u^-w_1$$
$$\text{-}y_1^+y_2^+\dots y_{2k}^+c^+v^+b^+d^-x_1^-w_6v_1^-w_1v_2^-w_2v_3^-w_3v_4^-w_4y_{6k+4}^-a^-$$

♣

2.3.1 $m_2 > 1$

We will work on diagrams with different parametrization as it gives simpler description than the original parametrization.

Let the E-data is given by

$$x_1^+x_2^+\dots x_q^+\dots x_{p_1}^+P_1y_1^+y_2^+\dots y_q^+\dots y_{p_2}^+P_2 \tag{2.3.1}$$

where P_1, P_2 represents negative blocks, and the letters on their edges are

$$\begin{aligned} P_1 &= y_{p_2}^-\cdots y_q^- \\ P_2 &= x_{p_1}^-\cdots x_q^-. \end{aligned}$$

This class represents $M(0,0;q-1,p_1-q-1,q-1,p_2-q_2-1)$.

Theorem 10 *The E-datum $M(0,0;q-1,p_1-q-1,q-1,p_2-q_2-1)$ is realizable if and only if q does not divide p_1+p_2.* ♠

Proof Define two sequences of integers λ_k and λ'_k for non-negative integers k by

$$\begin{aligned} \lambda_{k+1} &\equiv \lambda'_k + p_2 \pmod q \\ \lambda'_{k+1} &\equiv \lambda_{k+1} + p_1. \end{aligned}$$

Let us denote by $\{x\}$, the least positive integer which is congruent to x modulo q. Then setting

$$\lambda_1 = p_2, \tag{2.3.2}$$

P_1 may be written as, omitting their types and signs,

$$\begin{aligned} P_1 &= y_{p_2} y_{p_2-q} \cdots y_{\{\lambda_1\}} x_{\{\lambda_1+p_1-q\}} x_{\{\lambda_1+p_1-2q\}} \cdots x_{\{\lambda'_1\}} \\ & \quad y_{\{\lambda'_1+p_2-q\}} y_{\{\lambda'_1+p_2-2q\}} \end{aligned}$$

For this block to have y_q^- in its end, λ'_m must be congruent to $-p_2$ for some m. For this is to be realized it is necessary and sufficient that x_q^- does not appear before y_q^- appears in the block. x_q^- appears if $\lambda_n \equiv -p_1$. We find by (2.3.1) with (2.3.2) that

$$\begin{aligned} \lambda'_m &\equiv m(p_1+p_2) \equiv -p_2 \\ \lambda_n &\equiv (n-1)p_1 + np_2 \equiv -p_1 \end{aligned}$$

We see that the second equation is equivalent to $n(p_1+p_2) \equiv 0$. Hence, as long as the modulus q does not divide p_1+p_2, we have $m < n$. Hence the negative block is realized.

2.4 Torsion Linking Form

We saw, most of the manifolds we have listed may be distinguished by inspecting their first homology groups. However, since any arithmetic progressions with distinct common differences intersect infinitely often, we cannot but have unboundedly many pairs of homologically identical manifolds.

2.4.1 Definition

To obtain a 'homologically well-defined' torsion linking form, one has to have the common period for the. Thus we are lead to the local linking forms.

Definition 13 ([8]) *Let M be a closed orientable 3-manifold. Let x and y be 1-cycles in M and let n be an integer such that nx bounds a 2-complex $c(x)$. Then the torsion linking form at denoted as $\mathrm{lk}_M : H_1(M;\mathbb{Z}) \times H_1(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ is defined by*

$$\mathrm{lk}_M([x],[y]) := \frac{c(x)\cdot y}{n}.$$

Since (torsion) linking form is a symmetric bi-linear form[6], if written in a matrix form, the base change induce symplectic transformations: Let lk_b be the torsion linking form with respect to a base b. By an invertible base change $b = Bb'$, linking form is transformed to be

$$\mathrm{lk}_{b'} = {}^tB(\mathrm{lk}_b)B \tag{2.4.1}$$

However we will present linking forms in a row verctor in what follows.

Example 16 ($L(p,q)$) Let $L(p,q) = V_1 \cup V_2$ be the standard construction by two solid tori.[3] Let $D_i \subset V_i$ denote a disk which the meridian m_i bounds. Let us calculate the linking form of the generator of the homology group and itself. One representative element is l_1, whence

$$\mathrm{lk}(l_1,l_1) = \frac{(-qD_1 \cup D_2)\cdot l_1}{p} = -\frac{q}{p}.$$

Homologous to l_1 is $-ql_2 + sm_2$, which when multiplied by p bounds D_2.[4] So the self-linking form of rl_2 is

$$\mathrm{lk} = \frac{((-qD_1)\cup(1-ps)D_2)\cdot(-ql_2)}{p} = \frac{(1-ps)(-q)}{p} \equiv -\frac{q}{p}.$$

[3]The so called Lens spaces are those which obtained by gluing two solid tori (handle bodies of genus 1) along their boundaries.

Definition 14 *Let V_1 and V_2 be two copies of an oriented solid tori, and h a homeomorphism from ∂V_2 to ∂V_1 which takes a meridian of ∂V_2 to a (p,q)-curve on ∂V_1. Then, the lens space $L(p,q)$ is (any homeomorphic image of) a sum of V_1 and V_2 whose boundaries are identified with each other by h such that two orientations of ∂V_1 and ∂V_2 are compatible; This specified construction is often denoted as*

$$L(p,q) := V_1 \underset{h}{\cup} V_2.$$

[4]By the convention explained in the preceding footnote, the attaching homeomorphism $h : \partial V_2 \to \partial V_1$ must be orientation-reversing. Then compare with the calculation carried out in the proof of Proposition 4.

However, the linking form is not an algebraic invariant. For any p' with $\mathrm{gcm}(p,p')=1$, $p'l_1$ is also a generator of H_1, whose self-linking is

$$\mathrm{lk} = \frac{(-p'qD_1 \cup p'D_2)\cdot p'l_1}{p} = -\frac{p'^2 q}{p}.$$

♣

As is seen by this example, torsion linking form of itself is not an invariant but the set of values of linking forms of a specified pair of cycles is trivially an invariant. And in the above case, it is precisely the set of quadratic residue or quadratic non-residue, depending on p being the residue or not.

In general, the homology groups of genus 2 manifolds have two summands. So in applying torsion linking form, we have to know the automorphism groups of abelian groups (of at most two summands).

Proposition 6 *Let $d \le m < n$, where d denotes* $\mathrm{gcm}(m,n)$.

$$\mathrm{Aut}(\mathbb{Z}_m \oplus \mathbb{Z}_n) \cong \begin{cases} GL(2,\mathbb{Z}_m) & \textit{if } \ m=n \\ \mathbb{Z}_m^\times \times \mathbb{Z}_n^\times & \quad d=1 \\ (\mathbb{Z}_m^\times \oplus \mathbb{Z}_d) \times (\mathbb{Z}_n^\times \oplus \mathbb{Z}_d) & \quad d>1 \end{cases} . \tag{2.4.2}$$

Proof Consult [3]. □

2.4.2 Demonstrations

As we have remarked in §, $M(0,0;4(7i+1),0,4(7i+1),2)$ and $M(0,0;4(9i+1)+2,0,4(9i+1)+2,2)$, $i=0,1,2,\ldots.$ have the same first homology groups. So,

Question 1 *Are $M(0,0;4k,0,4k,0)$ and $M(0,0;4l+2,0,4l+2,0)$, $k=7i+1, l=9i+1$, different manifolds?*

Let us study this problem by torsion linking form.

Example 17 (Determining the torsion linking form) Let $M = M(0,0;4(7i+1),0,4(7i+1),2)$. An E-datum for M is

$$\begin{aligned}
&x_1^+x_2^+\ldots x_{4k}^+d^+\\
&\text{-}b^-y_3^-x_4^-\cdots y_{4k-1}^-x_{4k}^-v_2^-y_2^-x_3^-\cdots y_{4k-2}^-x_{4k-1}^-v_1^-y_1^-x_2^-\cdots y_{4k-3}^-x_{4k-2}^-c^-\\
&\text{-}\underbrace{y_1^+y_2^+\ldots y_{3k}^+c^+v_1^+v_2^+b^+}_{\text{positive block}}d^-x_1^-y_4^-\ldots x_{4k-3}^-y_{4k}^-a^-.
\end{aligned}$$

By the decomposition, Let D_{21} denote the disk which the first negative block

$$b^-y_3^-x_4^-\cdots y_{4k-1}^-x_{4k}^-v_2^-y_2^-x_3^-\cdots y_{4k-2}^-x_{4k-1}^-v_1^-y_1^-x_2^-\cdots y_{4k-3}^-x_{4k-2}^-c^-$$

and the arc CB (drawn in the figure below) bounds. Similarly, the disk that the other negative block and the arc AD bounds will be called D_{22}. We assume

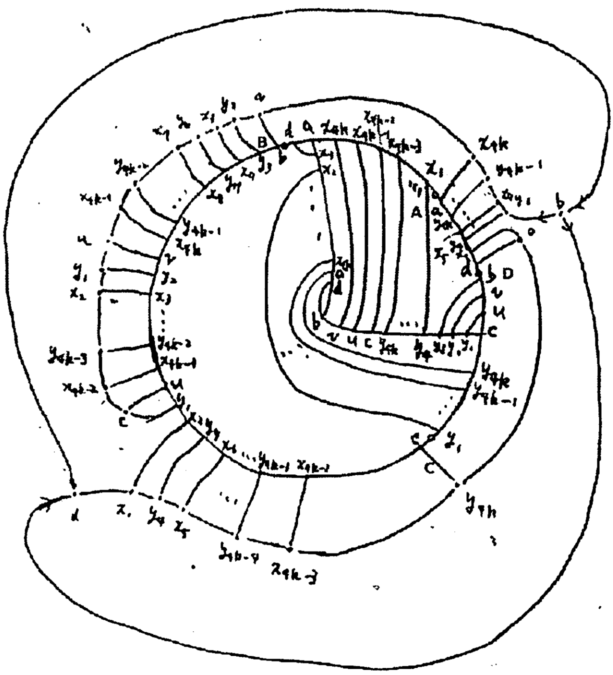


Figure 2.6: Notice the new vertices A, B, C, D written in capital letters.

that D_{2i}'s are attached so that the straight (not winding) line segment on the diagram which connects two b's, and two d's are 'preferred longitudes' of each handle. Note that this does not affect the value of the torsion linking form. Let us use the following notation to denote 2-complexes:

$$(a, b, c, d) := aD_{11} \cup bD_{12} \cup cD_{21} \cup dD_{22}.$$

Tracing the boundary curve of D_{21} it is observed that it winds about each handle only when it crosses ∂D_{11} (resp. ∂D_{12}) at the vertex which sits behind a^+ (resp. c^+), with respect to the orientation of the E-cycle. In fact, ∂D_{21} winds about the $(1,1)$-handle $3k$ times and $(1,2)$-handle $3k+1$ times.

$$b^- y_3^- x_4^- \cdots y_{4k-1}^- x_{4k}^- v_2^- y_2^- x_3^- \cdots y_{4k-2}^- x_{4k-1}^- v_1^- y_1^- x_2^- \cdots y_{4k-3}^- x_{4k-2}^- c^-$$

Therefore, ∂D_{21} bounds, inside the 'upper ball' $3k$ D_{11} and $3k+1$ D_{22}. Recall that the relators in the presentation

$$\pi_1(M) = \langle \alpha, \beta \mid r_1, r_2 \rangle$$

is

$$r_1 = (\beta(\beta\alpha)^k)^3\beta^2\alpha^{-1}$$
$$r_2 = \alpha(\alpha\beta)^k\alpha^2\beta^{-1}.$$

Let us write abelianized α, β as a, b, respectively. Then the homology type of ∂D_{21} is

$$[\partial D_{21}] = (3k-1)a + (3k+5)$$

Hence, we have

$$(3k-1)a + (3k+5)b = \partial(3k, 3k+1, 1, 0) \tag{2.4.3}$$

Now we can read ∂D_{22} directly from the E-datum.

$$\text{-}d^- x_1^- y_4^- \cdots x_{4l-3}^- y_{4l}^- x_{4l+1}^- v_1^- y_1^- x_2^- \cdots y_{4l+1}^- x_{4l+2}^-$$
$$\text{-}v_2^- y_2^- x_3^- \cdots y_{4l-2}^- x_{4l-1}^- y_{4l+2}^- a^- .$$

Winding about D_{11} handle (resp. D_{12}) occurs at the letters $-xy-$ (resp. $-yx-$ and AD). Hence, combining this with

$$[\partial D_{22}] = (k+3)a + (k-1)b \tag{2.4.4}$$

we have

$$(k+3)a + (k-1)b = \partial(k+1, k, 0, 1) \tag{2.4.5}$$

Collecting () and ()

$$(3k-1)a + (3k+5)b = \partial(3k+1, 3k, 1, 0) \tag{2.4.6}$$
$$(k+3)a + (k-1)b = \partial(k+1, k, 0, 1) \tag{2.4.7}$$

The generator element for $\mathbb{Z}_2$-part and $\mathbb{Z}_{9k+7}$-part is $5a - 4b =: a'$ and $-a + b =: b'$, respectively. In this bases,

$$2a' = \partial(3, -1, -1, 3) \tag{2.4.8}$$
$$(9k+7)b' = \partial(-2k-2, 2k+1, k+1, -3k-2) \tag{2.4.9}$$

$$
\begin{aligned}
\mathrm{lk}(a',a') &= \frac{1}{2}(3,-1,-1,3)\cdot(5a-4b) \\
&= \frac{1}{2}\{3\times 5+(-1)\times(-4)\} \\
&\equiv \frac{1}{2} \\
\mathrm{lk}(a',b') &= \frac{1}{2}(3,-1,-1,3)\cdot(-a+b) \\
&= \frac{1}{2}\{3\times(-1)+(-1)\times 1\} \\
&\equiv 0 \\
\mathrm{lk}(b',a') &= \frac{1}{9k+7}(-2k-2,2k+1,k+1,-3k-2)\cdot(5a-4b) \\
&= \frac{1}{9k+7}\{(-2k-2)\times 5+(2k+1)\times(-4)\} \\
&= \frac{1}{9k+7}(-18k-14) \\
&\equiv 0 \\
\mathrm{lk}(b',b') &= \frac{1}{9k+7}(-2k-2,2k+1,k+1,-3k-2)\cdot(-a+b) \\
&= \frac{1}{9k+7}\{(-2k-2)\times(-1)+(2k+1)\times 1\} \\
&= \frac{4k+3}{9k+7}
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\mathrm{lk}_M &:= (\mathrm{lk}(a',a'),\mathrm{lk}(a',b'),\mathrm{lk}(b',a'),\mathrm{lk}(b',b')) && (2.4.10)\\
&= (\frac{1}{2},0,0,\frac{4k+3}{9k+7}) && (2.4.11)
\end{aligned}
$$

In the same manner, for $N = M(0,0;4l+2,0,4l+2,2)$ we find

$$
\mathrm{lk}_N = (\frac{1}{2},0,0,\frac{4l+5}{7l+9}). \qquad (2.4.12)
$$

♣

Example 18 (Demonstration to distinguish manifolds) Let $M_i = M(0,0;4(7i+1),0,4(7i+1),2)$ and $N_i = M(0,0;4(9i+1)+2,0,4(9i+1)+2,2)$. Note that

$$
H_1(M_i) = H_1(N_i) = \mathbb{Z}_2 \oplus \mathbb{Z}_{63i+16}
$$

The torsion linking form is

$$
\begin{aligned}
\mathrm{lk}_{M_i} &= (\frac{1}{2},0,0,\frac{28i+7}{63i+16}) \\
\mathrm{lk}_{N_i} &= (\frac{1}{2},0,0,\frac{36i+9}{63i+16}).
\end{aligned}
$$

If there is a homeomorphism $f : N_i \to M_i$, then

$$\mathrm{lk}_{f(N_i)} = \mathrm{lk}_{M_i}.$$

Therefore (not equivalently though) instead we can argue with the automorphisms of H_1.

- If i is an even number, then $63i+16$ is also even, hence the two summands in H_1 has a g.c.m. 2. Therefore possible automorphisms of H_1 are

$$\begin{aligned} a &\mapsto a, a + (63j+8)b \\ b &\mapsto qb, qb + a \end{aligned}$$

with q an integer satisfying $(q, 63i+16) = 1$ and $j = i/2$. In particular, q must be odd. Comparing each component of lk, we can restrict the automorphism. So doing for the first component, with $\epsilon = 0$ or 1,

$$\begin{aligned} \frac{1}{2} &= \mathrm{lk}_{f(N_i)}(f_*(a), f_*(a)) \\ &= \mathrm{lk}_{N_i}(a + \epsilon(63j+8)b, a + \epsilon(63j+8)b) \\ &= \mathrm{lk}_{N_i}(a,a) + \epsilon(63j+8)\mathrm{lk}_{N_i}(a,b) + \epsilon(63j+8)\mathrm{lk}_{N_i}(b,a) + \epsilon^2(63j+8)^2\mathrm{lk}_{N_i}(b,b) \\ &= \frac{1}{2} + 0 + 0 + \epsilon^2(63j+8)^2\frac{36i+9}{63i+16} \\ &= \frac{1}{2} + \epsilon^2(63j+8)\frac{36i+9}{2} \\ &\equiv \frac{1}{2} + \frac{\epsilon^2 j}{2} \end{aligned}$$

Hence, especially we see that if j is odd, ϵ must be zero. Comparing the second (or equivalently the third) component,

$$\begin{aligned} 0 &= \mathrm{lk}_{f(N_i)}(f_*(a), f_*(b)) \\ &= \mathrm{lk}_{N_i}(a + \epsilon(63j+8)b, qb + \delta a) \\ &\equiv \frac{\epsilon + \delta}{2}, \end{aligned}$$

where δ is 0 or 1. Combining these, if j is odd, the possible automorphism is

$$(a,b) \mapsto (a, qb).$$

and if j is even

$$(a,b) \mapsto (a+, qb), (a + (63j+8)b, a + qb).$$

- If i is an odd number, then $63i+16$ is also odd. Therefore possible automorphisms is only

$$(a,b) \mapsto (a, qb) \tag{2.4.13}$$

where q an integer with $(q, 63i+16) = 1$. In particular, q can be an even number say 2, hence the comparing second components lk does not yield anything.

♣

Example 19 (Demonstration Continued) It still remains to compare the fourth component. If the automorphism is of type $(a,b) \mapsto (a,qb)$, then

$$\begin{aligned}\frac{28i+7}{63i+16} &= \mathrm{lk}_{f(N_i)}(f_*(b), f_*(b)) \\ &= q^2 \mathrm{lk}_{N_i}(b,b) \\ &= q^2\frac{36i+9}{63i+16}\end{aligned}$$

Thus,

$$\frac{28i+7}{36i+9}$$

must be a quadratic residue modulo $63i+16$. Let us first check the numerator and the denominator of this fraction has any common divisor with the modulus.

1. Let (p,q) denote the greatest common divisor of p and q. By the Euclidean algorithm, we see that

   $$(28i+7, 63i+16) = (4i+1, 63i+16) = (4i+1, 3i+1) = (i, 3i+1) = (i,1) = 1$$

   Hence, the numerator is relatively prime to the modulus.

2. For the denominator

   $$(36i+9, 63i+16) = (4i+1, 63i+16) = 1$$

   Hence, the denominator is also relatively prime to the modulus.
   It is desirable to determine the quadratic residuocity of each number.

3. However, in comparing the fourth component, it is sufficient for us to know whether $28i+7$ and $36i+9$ can be both residues or both non-residues, at the same time. Thanks to the fact these numbers are co-prime to the modulus, we can find the reciprocal number of each number by the Euclidean algorithm:

   $$\frac{63i+16}{28i+7} = 2 + \frac{1}{3+\frac{1}{1+\frac{1}{7i+1}}} \tag{2.4.14}$$

   and, one solution to $x-(7i+1)y=-1$ is $(x,y)=(7i,1)$, hence

   $$2+\frac{1}{3+\frac{1}{1+\frac{1}{7i}}} = \frac{63i+7}{28i+3}$$

   Thus, $(28i+7)^{-1} \equiv 63i+7 \equiv -9$. In the same manner,

   $$(36i+9)\times 7 \equiv 36\cdot 7i + 63 - 4(63i+16) \equiv -1$$

Therefore inspecting whether the value

$$(28i+7)(36i+9)^{-1} \equiv -7(28i+7)$$

is a residue or not decide that the two numbers in consideration are both residues are non-residues. Moreover, since the quadratic residuocity is preserved if we invert the number, and for $-(28i+7)^{-1} \equiv 9 \equiv 3^2$, we just have to check if 7 is quadratic residue or not. This is of better practical use.

For example, when $i = 1$, $63i + 16 = 79$ is a prime. Applying quadratic reciprocity law, we find

$$\begin{aligned}\left(\frac{7}{79}\right) &= -\left(\frac{79}{7}\right)\\ &= -\left(\frac{2}{7}\right)\\ &= -1\end{aligned}$$

Hence we may conclude, $28i + 7 = 35$ and $36i + 9 = 45$ cannot be a quadratic residue at the same time, thus corresponding manifolds M_1 and N_1 are not homeomorphic.

i	$63i+16$	primeness	residuocity	square of
1	79	p	N	–
2	142	$2*71$	N	–
3	205	$5*41$	N	–
5	331	p	N	–
6	394	$2*197$	R	91^2
7	457	p	N	–
9	583	$11*53$	N	–
10	646	$2*17*19$	N	–
11	709	p	N	–
13	835	$5*167$	N	–
14	898	$2*449$	R	289^2
15	961	31^2	R	331^2
17	1087	p	N	–
18	1150	$2*5^2*23$	N	–
19	1213	p	R	475^2

Table 2.1: The table of the quadratic residuocity of 7 modulo $63i+16$, where R (or N) stands for "being a residue" (or not being a non-residue, respectively).

♣

2.4.3 Appendices

Here are some useful facts in computing quadratic residue symbol (Legendre symbol). (finite) continued fractions is an analytic expression of the Euclidean algorithm.

Proposition 7 *Let p, q be two integers. For $x, y \in R$, Suppose that the regular continued fraction expansion of p/q is given as follows;*

$$\begin{aligned} \frac{p}{q} &= b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{\cdots b_{n-1} + \frac{1}{b_n}}}} \\ &=: [b_0; b_1, b_2, \ldots, b_{n-1}, b_n] \end{aligned}$$

Then a solution to

$$px - qy = 1$$

is given by $(x, y) = (m, n)$, where m, n are computed by

$$p\overline{p} - qb_n = (-1)^n$$

$$[b_0; b_1, b_2, \ldots, b_{n-1}, \frac{1}{b_n}]$$

See [2].

Theorem 11 *Let p, q be two distinct odd primes. Then the following formula holds:*

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

♠

See [2] for the proof.

Proposition 8 *Let p, q be two distinct prime numbers, and a an integer. Then a is also a quadratic residue modulo pq if and only if a is a quadratic residue modulo each of p and q.*

Proof $(\mathbb{Z}_p \times \mathbb{Z}_q)^\times \cong \mathbb{Z}_{pq}^\times$. Let $a \equiv x^2 \pmod p, \equiv y^2 \pmod q$. Then $(a, a) = (x^2, y^2) = (x, y)^2$, which is a square element in $\mathbb{Z}_{pq}^\times$. □

Remark 8 Elementary and effective proof: We only have to find k which satisfies

$$\begin{aligned} k &\equiv x \pmod p \\ k &\equiv y \pmod q. \end{aligned}$$

$$pa - qb = y - x$$

It always have a solution since p and q are relatively prime, and there is an effective algorithm to find it.

Chapter 3

Geometric Properties

3.1 Exploring Klein bottles

Let us begin with $M(0,0;2,0,2,0)$.[1]

Example 20 Recall that its first homology group is $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. Hence there is a cycle which bounds when doubled, whence the existence of an embedded unorientable surface is expected.[2]
Indeed the manifold admits a Klein bottle as is seen by connecting the arcs 7-7-3-3-as in the diagram. One may also notice 2-8-8-2 also bounds a Klein bottle. We can make a 2-fold branched covering along this Klein bottle, the resulting manifold killing its $\mathbb{Z}_2$ part.

♣

We want to know if the existence of even order element in the homology groups implies whether an unorientable surface is embedded or not.

3.1.1 Method of DS-diagrams

DS-diagrams provide a suitable computational basis for the geometry of surfaces as well as curves.

Let M be a 3-manifold with a simple spine P. Any properly embedded closed surface F may be isotoped so that $F \cap (M - P)$ is a disjoint set of embedded disks. Hence the surface may be represented by the curve of intersection $F \cap P$.

Let $\gamma := F \cap P$ be a simple closed curve on the sphere, on which our diagram is supposed to lie. Let us think of some computable means to handle these geometry.

[1]The contents of this section was implied by Professor Ishi'i.

[2]Suppose γ is a cycle which bounds when doubled. Let γ' be obtained by slightly perturbating 2γ so as to make it a simple closed curve. Since we can spread a Möbius band inbetween, the surface it bounds, if any, must be unorientable.

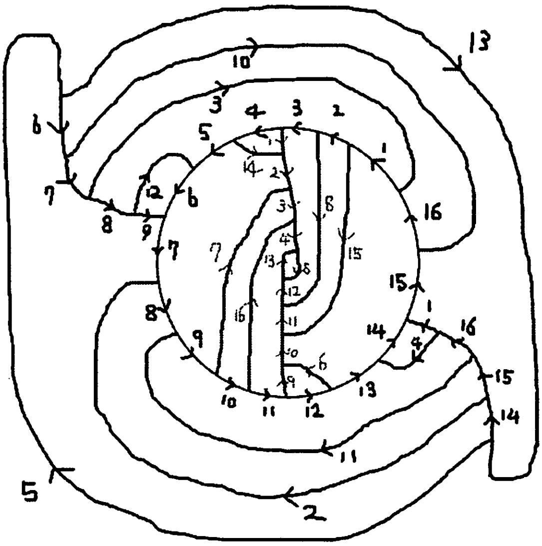


Figure 3.1: The diagram of $M(0,0;2,0,2,0)$.

One such way is to count the intersection of γ with the edges of the spine P. Let r_i (resp. c_i) denotes the number of $+$(resp. $-$) crossing points on the edge i. Let $p_i := r_i - c_i$, $q_i := r_i + c_i$. Then we consider, in a face σ

$$\sum_{i\in\partial\sigma} r_i = \sum_{i\in\partial\sigma} c_i \Leftrightarrow \sum_{i\in\partial\sigma} p_i = 0$$

Irreducibility requires what is more:

$$r_i = \sum_{i\in\partial\sigma} c_i \Leftrightarrow$$

The condition of being a closed curve is to be 0 modulo 2, that is for all faces σ

$$\sum_{i\in\sigma} q_i \equiv 0 \pmod 2). \tag{3.1.1}$$

Example 21 ($M(0,0;0,0,0,0)$) Let us consider the example by the simplest description using q's. We name edges with numbers from 1 to 8 as in the figure 3.3. The equation (1-0.1) becomes

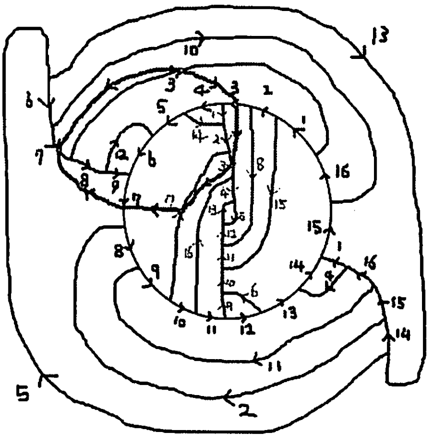


Figure 3.2: An embedded Klein bottle is drawn.

$$
\begin{array}{lr}
0. & q_3 + q_7 \equiv 0 \\
1. & 2q_1 + q_2 + q_3 + q_4 + q_6 + q_7 + q_8 \equiv 0 \\
2. & q_1 + q_2 + q_8 \equiv 0 \\
3. & q_2 + q_3 + q_4 + 2q_5 + q_6 + q_7 + q_8 \equiv 0 \\
4. & q_4 + q_5 + q_6 \equiv 0
\end{array} .
$$

Then essential restrictions are

$$
\begin{aligned}
q_1 + q_5 &= 0 \\
q_3 + q_7 &= 0
\end{aligned}
$$

with dependence relations

$$
\begin{aligned}
q_2 + q_8 &= q_1 \\
q_4 + q_6 &= q_5.
\end{aligned}
$$

The rank of this equation is 4. Let us write the solution as a row vector (q_1, q_2, q_3, q_4).

1. $(0,0,1,0)$, that is $q_1 = q_2 = q_4 = q_5 = q_6 = q_8 = 0$ and $q_3 = q_7 = 1$. Then the surface represented by the data $(0,0,1,0)$ obtained by natural inclusion into $\mathbb{Z}$ represents a torus.

2. $(0,1,0,0)$ is again a torus. We can streamline the argument by simply calculating the Euler characteristic. There are 2 edges which involves intersection, thus 2 vertices. There are 3 edges inside the diagram; those

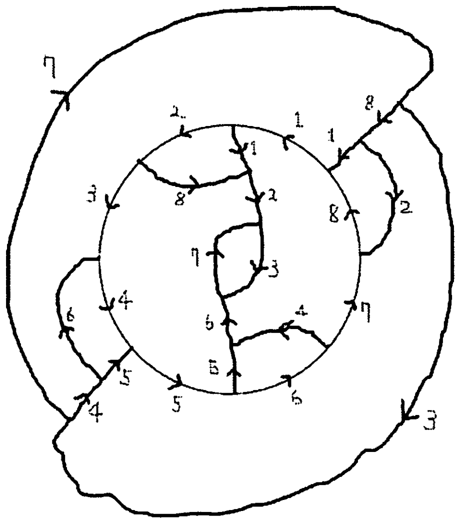


Figure 3.3: The DS-diagram of $M(0,0;0,0,0,0)$.

outside the E-cycle are attached to the these inside. And 1 face. So the Euler characteristic χ is

$$\chi = 2 - 3 + 1 = 0.$$

Hence the surface is either a torus or a Klein bottle. But we see that the surface has the diagram

$$ABCA^{-1}B^{-1}C^{-1}$$

hence is orientable. So the surface is a torus.

3. $(0,0,0,1)$ also represents a torus.

4. $(1,0,0,0)$ has a bit complicated structure. There are four cases according to the two undetermined crossings. Connecting arcs in any manner, there are 'equi-oriented' arcs, which shows that the diagram contains a Möbius band and that the surface is non-orientable.

♣

3.1.2 Back to the Klein bottles

We demonstrated in the previous section how surfaces in a manifold represented with a DS-diagram may be treated.

Note that we always have a trivial solution which represents a 3-ball, obtained by substituting 1 for all q_i, where an edge i is contained in some fixed maximal tree, and 0 for the other edges. Therefore to look for non-orientable surfaces efficiently, it is sufficient to look for solutions which is independent from this solution Since in $\mathbb{Z}_2$ sense,

Example 22 ($M(0,0;2,0,2,0)$) We first take a maximal tree of the spine; say, edges of index 1, 2, 4, 5, 9 and 12 (See Fig. 3.1). And evaluate associated q_i's as 0; $q_1 = q_2 = q_4 = q_5 = q_9 = q_{12} = 0$. Inspecting the central 2-gon yields

$$q_{13} = 0.$$

We may also find

$$q_6 = q_{14} = 0.$$

Other relations coming from the remaining regions are

$$\begin{aligned} q_8 + q_{11} + q_{15} &= 0 && (3.1.2) \\ 2q_3 + q_8 &= 0 && (3.1.3) \\ q_6 + 2q_7 + q_8 &= 0 && (3.1.4) \\ q_3 + q_7 + q_{16} &= 0 && (3.1.5) \\ 2q_{11} + q_{16} &= 0 && (3.1.6) \\ 2q_{15} + q_{16} &= 0. && (3.1.7) \end{aligned}$$

The solution to this system of equations is $q_3 = q_7$ and $q_{11} = q_{15}$. Thus one of the simplest (=less winding on the graph) Klein bottle looks like as in the figure 3.1; the red curve corresponds to the solution $q_3 = q_7 = 1$.

♣

In the following we assume $k \geq 2$, and all the equations should be understood in mod2.

Theorem 12 *Every manifold $M(0,0;2k,0,2k,0)$ contains a Klein bottle.* ♠

Proof Let the E-data of $M(0,0;2k,0,2k,0)$ be given by

$$x_1^+ \ldots x_{2k}^+ d^+ \underbrace{b^- y_1^- x_2^- y_3^- x_4^- \ldots y_{2k-1}^- x_{2k}^- c^-}_{\text{negative block}}$$

$$\text{-}y_1^+ y_2^+ \ldots y_{2k}^+ c^+ b^+ \underbrace{d^- x_1^- y_2^- x_3^- y_4^- \ldots x_{2k-1}^- y_{2k}^- a^-}_{\text{negative block}} .$$

We name the edge between a^- and x_1^+ as the edge 1, and x_1^+ and x_2^+ as the edge 2, and so on. Then we take the maximal tree as below;

$$\begin{aligned} q_1 = q_2 = \cdots = q_{2k} &= 0 \\ q_{2k+2} = q_{2k+3} &= 0 \\ q_{4k+5} = q_{4k+6} = \cdots = q_{6k+4} &= 0 \\ q_{6k+6} &= 0 \end{aligned} \tag{3.1.8}$$

First we have, from the assumption,

$$q_{2k+4} = q_{6k+7} = q_{6k+8} = 0 \tag{3.1.9}$$

Considering the balance equation for regions in the right half of the diagram,

$$2q_{6k+9} + q_{6k+10} + q_{6k+11} \cdots + q_{8k+8} = 0 \tag{3.1.10}$$

For those, rectangular regions

$$q_{6k+9} = q_{2k+6} = q_{6k+11} = q_{2k+8} = \cdots = q_{8k+5} = q_{4k+2} = q_{8k+7} \tag{3.1.11}$$

$$\begin{aligned} q_{8k+7} + q_{4k+4} + q_{6k+5} &= 0 \\ 2q_{2k+1} + q_{4k+4} &= 0 \end{aligned} \tag{3.1.12}$$

From which we find, setting $\alpha = q_{6k+9} = q_{2k+6} = \cdots$,

$$\begin{aligned} q_{6k+5} &= \alpha \\ q_{4k+4} &= 0. \end{aligned} \tag{3.1.13}$$

Likewise for the left half

$$\begin{aligned} 2q_{2k+5} + q_{2k+6} + q_{2k+7} + \cdots + q_{4k+3} &= 0 \\ q_{2k+5} = q_{6k+10} = q_{2k+7} = q_{6k+12} = \cdots = q_{4k+1} = q_{8k+6} &= q_{4k+3} \\ q_{4k+3} + q_{8k+8} + q_{2k+1} &= 0 \\ q_{8k+8} + 2q_{6k+5} &= 0 \end{aligned} \tag{3.1.14}$$

From which we find, setting $\beta = q_{6k+9} = q_{2k+6} = \cdots$,

$$\begin{aligned} q_{2k+1} &= \alpha \\ q_{8k+8} &= 0. \end{aligned} \tag{3.1.15}$$

Thus both the equations (3.1.10) and (3.1.14) reduces to the identical

$$(k-1)\alpha + (k-1)\beta = 0 \tag{3.1.16}$$

Therefore the dimension of the solution is 1 or 2, according to n even or odd;

$$\left.\begin{aligned}
q_1 = q_2 = \cdots = q_{2k} &= 0 \\
q_{2k+1} &= \beta \\
q_{2k+2} = q_{2k+3} = q_{2k+4} &= 0 \\
q_{2k+5} = q_{2k+7} = q_{2k+9} = \cdots = q_{4k+3} &= \beta \\
q_{2k+6} = q_{2k+8} = q_{2k+10} = \cdots = q_{4k+2} &= \alpha \\
q_{4k+4} &= 0 \\
q_{4k+5} = q_{4k+6} = \cdots = q_{6k+4} &= 0 \\
q_{6k+5} &= \alpha \\
q_{6k+6} = q_{6k+7} = q_{6k+8} &= 0 \\
q_{6k+9} &= \alpha \\
q_{6k+10} = q_{6k+12} = \cdots = q_{8k+6} &= \beta \\
q_{6k+11} = q_{6k+13} = \cdots = q_{8k+7} &= \alpha \\
q_{8k+8} &= 0
\end{aligned}\right\} \tag{3.1.17}$$

$$\begin{aligned}
q_{6k+9} = q_{2k+6} = q_{6k+11} = q_{2k+8} = \cdots = q_{8k+5} = q_{4k+2} &= q_{8k+7} \\
q_{8k+7} = q_{2k+3} &= 0 \\
q_{4k+5} = q_{4k+6} = \cdots = q_{6k+4} &= 0 \\
q_{6k+6} &= 0 \\
q_{6k+6} &= 0
\end{aligned} \tag{3.1.18}$$

To prove that the surface is Klein bottle, we just have to compute the Euler characteristic.
Unorientability may be checked by looking at two edges both of which connects with each other at the edge $6k+5$. Since they have the same orientation, the polygonal diagram it bounds contains a Möbius band, hence unorientable. □

3.2 Branched Covering

Let us first review the earlier remark 3 where we stated;

Example 23 $M(0,0;2,0,2,0)$ is a branched double cover of $L(4,3)$ branched along the red arc drawn in the figure 3.5. The branched knot downstairs

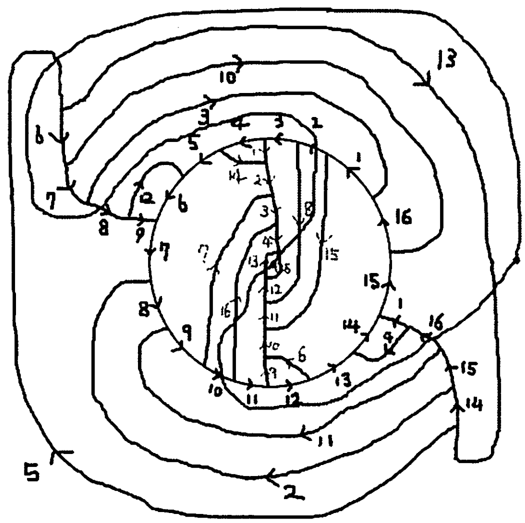


Figure 3.4: The double cover of the diagram in the Figure 3.5

represents the trivial element in the fundamental group, and thus a trivial knot in $L(4,3)$.[3] However, when pushed into one of the handlebodies of its Heegaard-splitting, it looks like as follows; Fig.3.6.

First, we will show that we may make the unknotted arc, which is in fact the branched set, 'float' to the surface of the diagram, and make it slip through the surface to lower into the northern hemisphere.

It is obvious that through non-singular region of the spine the arc may slip through, but through those singular points such as edges a suspicion may occur. But it turns out we may make it simply go through the edges in this case as well, as shown in the following series of schemes; The point is that the edges to be crossed all have the corresponding edges on the E-cycle, thus adjacent to the extra land adhered to the positive block. This extra part attached to the initial northern hemisphere, gives a space for the arc to slip through the edges.(Figure 3.9) Thus we simply get the knot in northern hemisphere as in figure 3.8, whose picture in a solid torus is figure 3.6 given earlier.

[3] i.e. homotopic to the zero map.

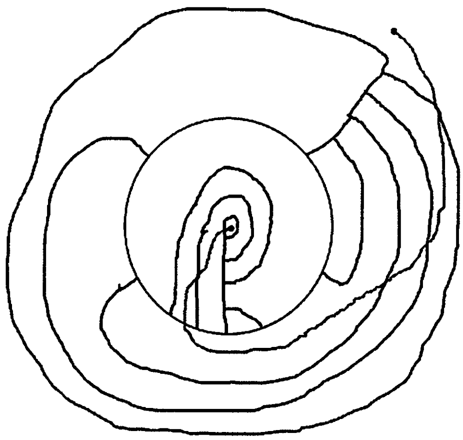


Figure 3.5: 'Quotient' diagram of Figure 3.4., with respect to the π-rotational action.

♣

In the same manner we obtain the following.

Theorem 13 *Let $p : M(0,0;2k,0,2k,0) \to L(2k+2,2k+1)$ be the branched covering explained above. The branched knot downstairs $\underline{br}_k$ may be expressed in the braid presentation as in the followings;*

$$\underline{br}_k = \left(\prod_1^{2k-1} b_i \prod_1^{2k} b_i \prod_2^{2k-1} b_i \right)^{-1} \prod_2^{2k} b_i^{-1},$$

where the braid is closed in a solid torus so that each strands runs parallel to the longitudinal curve. ♠

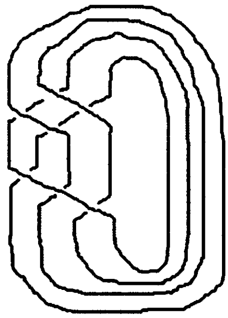


Figure 3.6: Branched knot expressed in a braid form. It lies in the solid torus such that each braid runs parallel to the longitudinal curves.

3.3 Fibered knots

Closely related to the topic in the previous section is the problem of fibered-ness. There has been known an efficient algorithm based on the following argument.[4]

Let K be a fibered knot in M, and let F_t denote the fiber over a point $e^{2\pi it} \in S^1$ of the base space. As in the preceding section, F_t may be isotoped so that $F_t \cap (M - P)$ consists only of some disks. Suppose further that F_t intersects the spine P transversally for $\forall t$.[5] Then each point of the spine P intersects a unique fiber, whose level value then gives a coordinate to P. So the layer pattern $\{F_t\}$ of fibers naturally gives a global (2-dimensional) coordinate to P by their level values and the tangents of the cross sectional curves.

Proposition 9 *Suppose K is a fibered knot in a rational homology sphere M, i.e. there exists continuous map f from $M - K$ to S^1. Then f induces a global coordinate for $S(P)$.*

Combining ideas from differential forms we can get an effective algorithm to construct a fibration or detect fiberedness of a knot complement.

Proof We regard M as a differentiable manifold. This does not lose generality, since any two differentiable manifolds are homeomorphic if and only if they are diffeomorphic. Let M be a rational homology sphere, i.e. $H_1(M;\mathbb{R}) = 0$. Then for any knot K we have $H_1(M - K;\mathbb{R}) \cong \mathbb{R}$. If K is a fibered knot, then there exists a continuous map $f_0 : M - K \to S^1$. This f_0 satisfies that its pull-back integral extended over a loop γ which represents a generator of $H_1(M - K;\mathbb{R})$

$$\int_\gamma d(f_0(x))$$

[4]This method was shown to the author by professor Ishii.

[5]Except in a small neighborhood of each vertex of $S(P)$, to be precise.

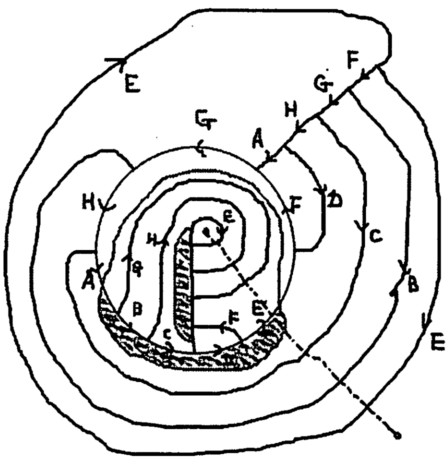


Figure 3.7: Branched knot, still inside the ball.

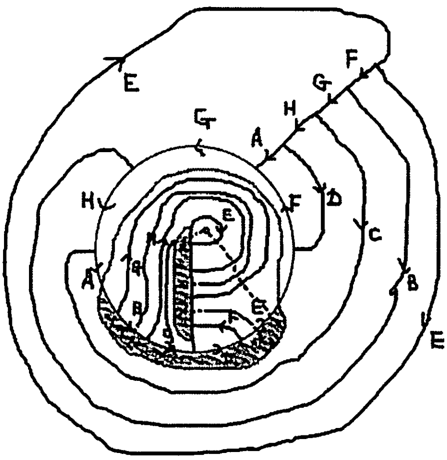


Figure 3.8: Branched knot risen up to the surface.

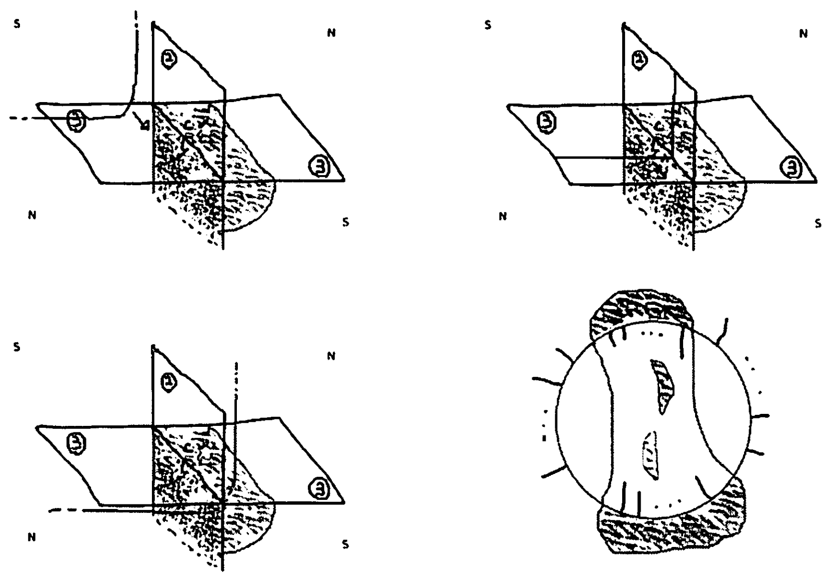


Figure 3.9: The orange part which sticks out in the southern hemisphere, is one of the 'reclaimed lands'(drawn in orange and blue in the right picture down) of the northern hemisphere

does not vanish. Such an integral represents an element of the cohomology group of the complement

$$H^1(M-K;\mathbb{R}),$$

or exactly, de Rham cohomology: whose precise definition is

$$H^1_{dR}(M-K;\mathbb{R}) \cong \{\omega \in \Omega^1(M-K); d\omega = 0\}/\sim,$$

where Ω^1 denotes the set of 1-forms and $\sim$ denotes the equivalence relation defined by

$$\omega_1 \sim \omega_2 \Leftrightarrow \exists\phi : M-K \to \mathbb{R} \text{ such that } \omega_1 - \omega_2 = d\phi.$$

And de Rham's theorem states this cohomology group in terms of differential forms is in fact isomorphic with the singular cohomology.

If we normalize each non-trivial form $\omega \in H^1(M-K;\mathbb{R})$ as

$$\int_\gamma \omega = 1$$

then for $\forall x \in M$

$$\exp(2\pi i \int_{x_0}^{x} \omega), \tag{3.3.1}$$

with a base point $x_0 \in M$, is well-defined on M, since its value does not depend on multiple circulation along γ.

Now, if we write $\Delta_\gamma \arg f$ for the variation in argument

$$\Delta_\gamma \arg f := \int_\gamma d(f(x))$$

then the equivalence relation $\sim$ among 1-forms is equivalent to

$$"f_1 \sim f_2 \Leftrightarrow \Delta_\gamma \arg f_1 = \Delta_\gamma \arg f_2"$$

Hence, as an representative element we may take those f as the monotonic one along certain loops. And the algorithm is based on the assumption that we take the edges of DS-diagrams for the "certain loops". □

Let us see how this algorithm may be effected on DS-diagrams in practice.

Example 24 ($L(2,1)$) Recall the irreducible diagram for $L(2,1)$. We name the edges in the diagram by numbers as in the figure 3.10.
Suppose we want to fibrate the complement of the knot A_0, obtained by connecting two 0-faces in the diagram in the figure 3.10. Let X be the complement of A_0 in $L(2,1)$, i.e. $X := L(2,1) - A_0$. We have seen that in order to construct a fiber it suffices to find a function f, defined on cycles $H_1(X)$ which satisfies

$$\int_\gamma df(x) \neq 0.$$

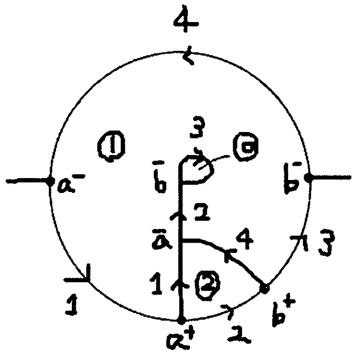


Figure 3.10: The inner part of the diagram of $L(2,1)$.

But by the monotonicity assumption, if we put

$$f(x) = \exp(2\pi i g(x)) \tag{3.3.2}$$

then $g(x)$ is a linear function for every edge of the diagram. Now let us call the increase in $g(x)$ along the edge i as a *pitch*, and denote it by a_i. In particular, a pitch of the edge 3 in the diagram is taken to be 1: $a_3 = 1$ (This is the non-trivial loop γ in the preceding argument). Other restrictions come from the fact that the function g is a homomorphism from $H_1(X)$ (or $\pi_1(X)$) to $\mathbb{R}$. Therefore, the sum of pitch along null-homologous cycle (null-homotopic loop) is 0. Let us apply this condition on the region 1 below, to determine the pitch of the edges on its boundary.

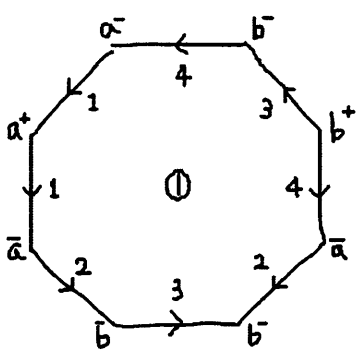


Figure 3.11: The vertices marked with triangle, resp. circle, is a local minimal point, resp. local maximal point.

$$a_1 + a_1 + a_2 + a_3 - a_2 - a_4 + a_3 + a_4 = 2a_1 + 2a_3 = 2 + 2a_1 = 0 \tag{3.3.3}$$

From the region 2, we have

$$-a_1 + a_2 + a_4 = 0 \tag{3.3.4}$$

Hence we have

$$\begin{aligned} a_1 &= -1 \\ a_2 + a_4 &= = -1 \end{aligned}$$

Taking

$$a_2 = a_4 = -\frac{1}{2}.$$

we can define a set of level values for all the edges, and by connecting the same level point with smooth curves, we see that the boundaries of any face have only one maximal point. Hence, the resulting flow is a gradient flow.

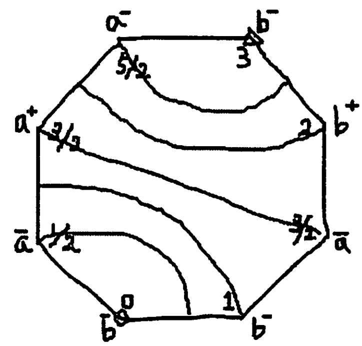


Figure 3.12: The vertices marked with triangle, resp. circle, is the local minimal point, resp. maximal point.

♣

Example 25 ($L(4,3)$) Let us take the $a_i (i = 1, 2, 3, 4)$ as basis for the solution. The equations reads

$$\begin{array}{lrcl} 1. & 2a_7 + a_8 & = & -a_1 + a_2 - a_5 \\ 2. & a_7 - a_8 & = & -a_2 + a_3 \\ 3. & a_8 & = & a_1 + a_2 + 2a_3 + a_5 \\ 0. & a_8 & = & a_1 + a_2 + 2a_3 + a_5 \end{array}$$

If we set $a_1 = x, a_2 = y, a_3 = z, a_4 = a$, $x + y + z + 1 = 0$. The values of the other edges are

$$a_6 = x - a \tag{3.3.5}$$

$$a_7 = x + 2y + z + 1 = y \tag{3.3.6}$$

$$a_8 = x + y + 2z + 1 = z \tag{3.3.7}$$

In choosing specific values for (x, y, z), we should bear in mind that, it is better the disk less undulating. So doing, we only have to keep track of the sign changes; One choice would be

edge	*value*	*sign*	*edge*	*value*	*sign*
3	z	$-$	1	x	$-$
1	x	$-$	7	y	$-$
2	y	$-$	-2	$-y$	$+$
3	z	$-$	-6	$a-x$	$+$
4	a	$-$	5	1	$+$
5	1	$+$	6	$x-a$	$-$
-4	$-a$	$+$	7	y	$-$
-8	$-z$	$+$	8	z	$-$

Table 3.1: The table of sign changes of each pitch of the edges in the region 3 (left) and 1 (right).

$$(x, y, z, a) = (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{4}).$$

The resulting curve on the diagram has 4 vertices, and 6 edges including the

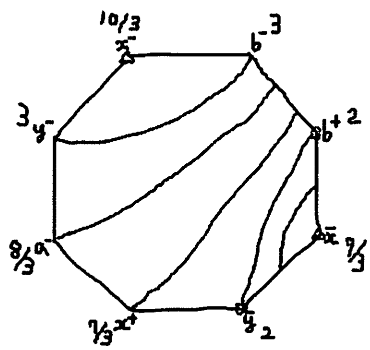


Figure 3.13: The level curves drawn in the region 1. The cross indicates a singular point.

arc A_0 inside the ball, and one face. Hence

$$\chi = 4 - 6 + 1 = -1$$

Therefore the fiber surface is found to be a punctured torus T^2.

♣

3.3.1 A series of fibered knots in lens spaces

For what we want to show in this section, let us simply observe the simple parametrization for the pitch in the DS-diagram of $L(6,5)$.

Example 26 ($L(6,5)$) Let the irreducible E-data of $L(6,5)$ be given by (theorem 6)

$$x_1^+ x_2^+ x_3^+ x_4^+ x_5^+ x_6^+ x_6^- x_1^- x_2^- x_3^- x_4^- x_5^- .$$

Let us assign numbers 1 to 12, to the edges on the E-cycle. The edge between x_5^- and x_1^+ will be assigned 1, and x_1^+ and x_2^+ assigned 2, and so on along the E-cycle. Let a_i denote the pitch of the edge i. It turns out that

$$\begin{aligned}
a_1 &= -1 - x - y - z - u \\
a_2 &= x \\
a_3 &= y \\
a_4 &= z \\
a_5 &= u \\
a_6 &= v \\
a_7 &= 1 \\
a_8 &= -1 - x - y - z - u - v \\
a_9 &= x \\
a_{10} &= y \\
a_{11} &= z \\
a_{12} &= u
\end{aligned} \tag{3.3.8}$$

Now the The cyclic character of this parametrization may be considered to reflect a certain relation with the automorphism group of $L(6,5)$, or equivalently its fundamental group.

♣

Let the lens space $L(2k+2, 2k+1)$ be given by the DS-diagram (B, G, f), presented in the figure 1.11 (§1, chapter 1), and let A_0 denote an unknotted arc in B which connects two faces with index 0.

Theorem 14 $L(2k+2, 2k+1) - A_0$ *are all fibered spaces.* ♠

Proof Let indices for the edges and the faces of diagrams be given as in the figure 1.11. Then we have the following equations for the set of pitch as follows: in it $m = \{x\}$ means that $p < m \leq 2p$ with $m \equiv x \pmod p$.

- From the region 1,

$$a_{\{-\bar{q}+2\}} + \sum_{k=p+3}^{2p} a_k = -(a_1 - a_{p-q+1} + a_{p+1}) \tag{3.3.9}$$

- From the region $l(l = 2, 3, \ldots, q-1)$;

$$a_{\{-l\bar{q}+2\}} - a_{\{-(l-1)\bar{q}+2\}} = a_{p-q+l} - a_l \tag{3.3.10}$$

- From the region q;

$$a_{\{-(q-1)\bar{q}+2\}} = a_q + \sum_{k=1}^{p-1} a_k + a_{p+1} \tag{3.3.11}$$

- From the region $q+1$;

$$a_{\{-(q+1)\bar{q}+2\}} = a_1 - a_{q+1} \tag{3.3.12}$$

- From the region $l(l = q+2, q+3, \ldots, p)$;

$$a_{\{-l\bar{q}+2\}} - a_{\{-(l-1)\bar{q}+2\}} = a_{l-q} - a_l \tag{3.3.13}$$

Now set $q = p - 1$, since this is the case which we need, Solving these we find that Substitute these into the equation (), we find that the sole restriction is

$$\sum_{k=1}^{p-1} a_k = 0. \tag{3.3.14}$$

We may specify the sign changes of the set of pitch as in the following table. Then we have

edge	*value*	*sign*
$q = p-1$	a_q	$-$
1	a_1	$-$
2	a_2	$-$
$\vdots$	$\vdots$	$\vdots$
p	a_p	$-$
$p+1$	1	$+$
$-p$	$-a_p$	$+$
$-\{-(q-1)\bar{q}+2\}$	$-a_{\{-(q-1)\bar{q}+2\}-p-1}$	$+$

Table 3.2: $-i$ in the 'edge' column means that "$-a_i$", a_i with reversed orientation.

The only relations among the variables are

$$\begin{aligned} 1 + \sum_{i=1}^{p-2} a_i &> 0 \\ 1 + \sum_{i=1}^{p-1} a_i &> 0 \end{aligned}$$

edge	*value*	*sign*
1	a_1	$-$
$\{-\bar{q}+2\}$	$a_{\{-\bar{q}+2\}-p-1}$	$-$
-2	$-a_2$	$+$
$-(p+2)$	$-(a_p - a_1)$	$+$
$p+1$	a_p	$+$
$p+2$	$a_p - a_1$	$-$
$p+3$	a_2	$-$
$p+4$	a_3	$-$
$\vdots$	$\vdots$	$\vdots$
$2p$	a_{p-1}	$-$

Table 3.3: $-i$ in the 'edge' column means that "$-a_i$".

Hence, we may take, for example

$$a_2 = a_3 = \cdots = a_{p-1} = -\frac{1}{p}. \tag{3.3.15}$$

In these values, to determine the genus of this fibered knot, we trace a level curve at the level $\frac{p-2}{p} + \epsilon$ ($\epsilon > 0$ sufficiently small) starting from 0-th face in the north. The level curve starting from the 1-gon at the north pole first crosses

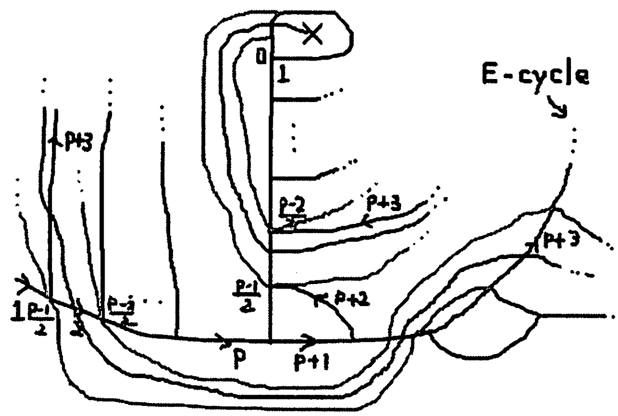


Figure 3.14:

the edge $p+1$, and then crosses the edge 2, since the level value decrease by $\frac{1}{p}$ along the edge 2 from $\frac{p-1}{p}$. Then it goes around the region 1 and crosses edge $p+3$ to enter the region 2. There again crosses the edge 2 on the E-cycle to go into the region $p-1$ outside the E-cycle. Notice that this curve is sandwiched between the level curve of $(p-1)/p$ and of $(p-2)/p$, which passes the vertex x_1 and x_2, respectively. It turns its way to come in to the region 1 again crossing the edge 7 on the E-cycle, and since the all the level curves whose level values are in the range $[\frac{p-2}{p}, \frac{p-1}{p}]$ injects into the edge 9, of level values cro The rest of the pass is determined by its track in each region inside the E-cycle. By this, we see that it intersects with E-cycle 3 times, and the number of the edges of the path inside the E-cycle is 5. Notice that as the boundary of the knot A_0, there is one point in the 0-face, and also one edge A_0 interior to the ball of the DS-diagram. Thus, the Euler characteristic is

$$(3+1) - (5+1) + 1 = -1$$

If we attach a disk to the knot A_0, then the Euler characteristic will increase by 1, hence a torus. □

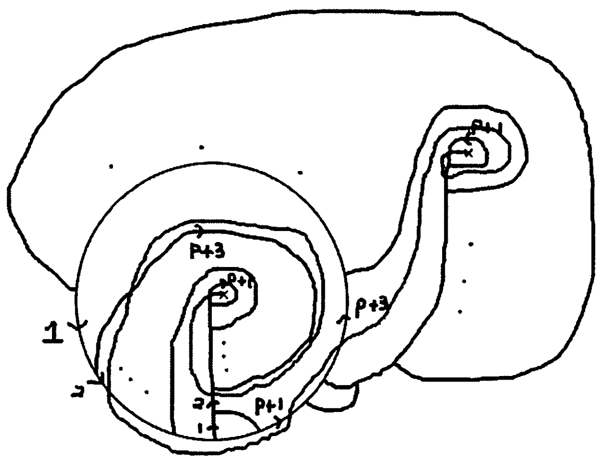


Figure 3.15: The level surface at the level value $\frac{p-3}{p}$ is drawn in red.

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First thanks goes to my supervisor professor Ishi'i. I believe I am the dumbest (graduate) student he has ever had, and for this I urge myself to repay his favor, kindness and patience, in some other way than in academic achievement.

What I wish is that this little exposition, which I endeavored to make it comical and enjoyable as much as possible, convey a sense of a joy of research. Joy of research, a curiosity for what is undiscovered, the passion for quest — are, as I was told, what I do lack. Now I consider one of the greatest harvests in preparing this paper is that even such person as I could dip into these intellectual desires, a short while though.

I strongly believe, DS-diagrams and E-data are noble notions, and notable Japanese product, that <u>you</u> should take a look, and deserve more attention. In particular, it will be far superior an educational stuff in topology class than the usual simplicial-complex one with triangulations.

Lastly I also want to thank everybody in 14-632B, who made my graduate life enjoyable. I also want to acknowledge my appreciation for my family, my sister in particular, for making a lunch almost everyday. Without this constant cheering, this little paper would not have appeared.